On the distribution of zeros of $\zeta(s)$

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Introduction

Let $\zeta(s)$ be the Riemann zeta-function and $s = \sigma + it$. We know

- Dirichlet series: $\zeta(s) = \sum_{n\geq 1} n^{-s}$ when $\sigma > 1$,
- Functional equation: $\zeta(s) = \chi(s)\zeta(1-s)$, where

$$\chi(s) = 2(2\pi)^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s).$$

- Trivial zeros: $\forall m \in \mathbb{N}, \zeta(-2m) = 0$.
- The nontrivial zeros $\rho = \beta + i\gamma$ satisfy $0 \leq \beta \leq 1$. (PNT: $0 < \beta < 1$).
- Riemann hypothesis: If $\rho = \beta + i\gamma$ is a nontrivial zero, then $\beta = \frac{1}{2}$.

Counting zeros

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The region $0 \le \sigma \le 1$ is called the **critical strip**, and the line $\sigma = \frac{1}{2}$ is known as the **critical line**. We will discuss

1. Counting nontrivial zeros: $N(T) = \#\{\rho : 0 < \gamma \le T\}$,
2. Zeros off the critical line: $N(\sigma, T) = \#\{\rho : \beta \ge \sigma, 0 < \gamma \le T\}$,
3. Zeros on the critical line: $N_0(T) = \#\{\rho : \beta = \frac{1}{2}, 0 < \gamma \le T\}$.

Highlights

We will go over the following classical results:

- von Mangoldt (1905): $N(T) \sim (2\pi)^{-1} T \log T$,
- Bohr–Landau (1914): $N(\sigma, T) \ll_\sigma T$ for fixed $\sigma > \frac{1}{2}$.
- Hardy–Littlewood (1921): $N_0(T) \gg T$.
- Selberg (1942): $N_0(T) \gg N(T)$, or a positive proportion of ρ satisfies RH.

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Riemann ξ-function

Define

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Then

- $\xi(s)$ is entire.
- $\xi(s) = \xi(1-s)$.
- The zeros of $\xi(s)$ coincide with the nontrivial zeros of $\zeta(s)$.

$N(T) \longrightarrow \infty$

Recall complex function theory:

Definition (Entire function of finite order)

An entire function $f(z)$ is of order $0 \leq \lambda < \infty$ if

$$|f(z)| \ll_\varepsilon e^{|z|^{\lambda+\varepsilon}}, \quad z \longrightarrow \infty.$$

Theorem (Classification of nonzero entire functions of finite order)

If $f(z)$ is an entire function of order λ without zeros, then $f(z) = e^{p(z)}$, where $p(z)$ is a polynomial of degree $\leq \lambda$.

Proof idea

Theorem

If $f(z)$ is an entire function of order λ without zeros, then $f(z) = e^{p(z)}$, where $p(z)$ is a polynomial of degree $\leq \lambda$.

Corollary

If $f(z)$ is an entire function of order $n \in \mathbb{N}$ with finitely many zeros, then $|f(z)| \ll e^{C|z|^n}$.

By the theorem, $f(z) = e^{p(z)}q(z)$ for polynomial p, q s.t. $\deg p \leq n$, so

$$|f(z)| \ll (1 + |z|)^{\deg q} e^{C_0|z|^{\deg p}} \ll e^{C|z|^n}.$$

We will show that $\xi(s)$ is of order 1 but $\log |\xi(s)|$ is not $\leq C|s|$.

Proof of $N(T) \to \infty$

By very tricky integral transforms, it can be proved that

$$\xi\left(\frac{1}{2}+z\right) = \sum_{n\geq 0} c_n z^{2n}, \quad c_n \geq 0.$$

Thus, we have for $|z| = R \geq 2$ that

$$\left|\xi\left(\frac{1}{2}+z\right)\right| \leq \xi\left(\frac{1}{2}+R\right) \asymp R^2 \pi^{-\frac{R}{2}} \Gamma\left(\frac{R}{2}-\frac{1}{4}\right) = e^{\frac{1}{2}R\log R + O(R)},$$

so $\xi(s)$ is of order 1.

If $\xi(s)$ only has finitely many zeros, then $\xi\left(\frac{1}{2}+R\right) \leq e^{AR}$, which is a contradiction.

Asymptotic formula for $N(T)$

Theorem (von Mangoldt)

We have for $T \to +\infty$,

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Riemann (1859) stated it, but the first proof was due to von Mangoldt.

Asymptotic formula for $N(T)$

Because $\xi(s)$ has no real zeros,

$$2N(T) = \frac{1}{2\pi}\Delta_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} \arg \xi(s).$$

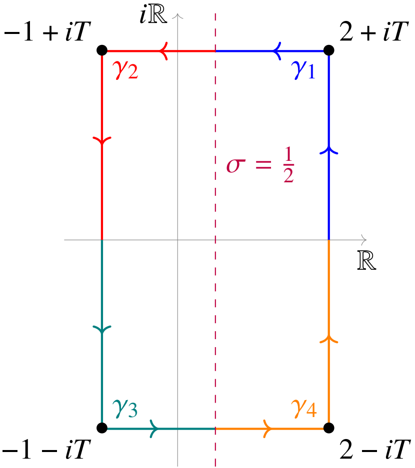
By $\xi(s) = \xi(1-s)$ and $\xi(\overline{s}) = \overline{\xi(s)}$,

$$\Delta_{\gamma_2} \arg \xi(s) = \Delta_{\gamma_4} \arg \xi(s),$$

$$\Delta_{\gamma_3} \arg \xi(s) = \Delta_{\gamma_1} \arg \xi(s),$$

$$\Delta_{\gamma_4} \arg \xi(s) = \Delta_{\gamma_1} \arg \xi(s).$$

$$\Rightarrow N(T) = \frac{1}{\pi}\Delta_{\gamma_1} \arg \xi(s).$$



Decomposition of $\arg \xi(s)$

$$N(T) = \frac{1}{\pi}\Delta_{\gamma_1} \arg \xi(s) = \frac{1}{\pi}\Delta_{\gamma_1} \arg \left[\frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right] + \underbrace{\frac{1}{\pi}\Delta_{\gamma_1} \arg \zeta(s)}_{S(T)},$$

where

$$\Delta_{\gamma_1} \arg[s(s-1)] = \arg\left(-\frac{1}{4} - T^2\right) = \pi, \quad \Delta_{\gamma_1} \arg(\pi^{-\frac{s}{2}}) = -\frac{T}{2}\log \pi.$$

By Stirling,

$$\Delta_{\gamma_1} \arg \Gamma\left(\frac{s}{2}\right) = \arg \Gamma\left(\frac{\frac{1}{2}+iT}{2}\right) = \frac{T}{2}\log\frac{T}{2} - \frac{\pi}{8} - \frac{T}{2} + O\left(\frac{1}{T}\right).$$

Main term for $N(T)$

Combining these results, we have

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right),$$

where

$$\begin{aligned} S(T) &= \frac{1}{\pi}\Delta_{\gamma_1}\arg\zeta(s) \\ &= \frac{1}{\pi}\Delta_{2}^{2+iT}\arg\zeta(s) + \frac{1}{\pi}\Delta_{2+iT}^{\frac{1}{2}+iT}\arg\zeta(s). \end{aligned}$$

Bounding $S(T)$

Theorem (von Mangoldt, 1905; Backlund, 1918)

As $T \longrightarrow +\infty$, $S(T) = O(\log T)$.

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When $\sigma = 2$,

$$\Re\zeta(s) = 1 + \sum_{n\geq 2}\frac{\cos(t\log n)}{n^2} \geq 1 - \sum_{n\geq 2}\frac{1}{n^2}$$
$$\geq 1 - \int_2^{+\infty}\frac{\mathrm{d}x}{x^2} = 1 - \frac{1}{2} = \frac{1}{2} > 0,$$

so $|\arg\zeta(s)| < \frac{\pi}{2}$ on $\sigma = 2$, which means

$$|\Delta_2^{2+iT}\arg\zeta(s)| = |\arg\zeta(2+iT) - 0| < \frac{\pi}{2}.$$

Bounding $S(T)$

For $\Delta_{2+iT}^{\frac{1}{2}+iT} \arg \zeta(s)$, if $\alpha < \beta$ are successive real zeros of $\eta(\sigma) = \Re\zeta(\sigma + iT)$ in $(\frac{1}{2}, 2)$, then $\eta(\sigma)$ does not change sign in $[\alpha, \beta]$, so

$$|\Delta_{\beta+iT}^{\alpha+iT} \arg \zeta(s)| \leq \pi.$$

Let $q(T) = \#$ of zeros of $\eta(\sigma)$ in $(\frac{1}{2}, 2)$. Then

$$|\Delta_{2+iT}^{\frac{1}{2}+iT} \arg \zeta(s)| \leq (q(T) + 1)\pi,$$

so one has

$$|S(T)| \leq q(T) + \frac{3}{2}.$$

It suffices to prove $q(T) = O(\log T)$.

Bounding $q(T)$

Theorem (Jensen)

Let f be analytic in $|z| \leq R$ s.t. $f(0) \neq 0$ and

$$n_f(r) = \#\text{ of zeros of } f(z) \text{ in } |z| \leq r.$$

Then

$$\int_0^R \frac{n_f(r)}{r} \mathrm{d}r = \int_0^1 \log|f(Re^{2\pi i x})| \mathrm{d}x - \log|f(0)|.$$

Let $f(z) = \zeta(z+2+iT) + \zeta(z+2-iT)$, so it follows from $f(z) \ll T^A$ in $|z| \leq 4$ and $f(0) \geq 4 - 2\zeta(2) > 0$ that

$$q(T) \leq n_f(2) \leq 4 \int_2^4 \frac{n_f(r)}{r} \mathrm{d}r \ll \log T.$$

Conclusion

Theorem (Riemann–von Mangoldt)

Let $N(T) = \#$ *of* ρ *with* $0 < \gamma \le T$. *Then*

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

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Littlewood's formula

Theorem (Littlewood, 1924)

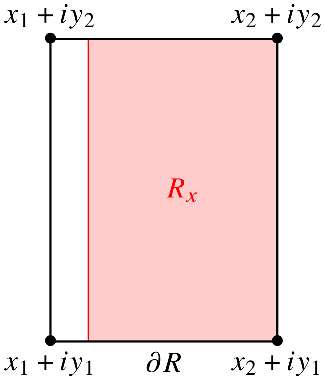
Let f be analytic in R and nonzero on ∂R,

$$R_x = \{z \in R : \Re(z) \geq x\},$$

$$\nu(x) = \#\text{ of zeros of } f \text{ in } R_x.$$

Then

$$2\pi \int_{x_1}^{x_2} \nu(x)\mathrm{d}x = \int_{y_1}^{y_2} \log\left|\frac{f(x_1+iv)}{f(x_2+iv)}\right| \mathrm{d}v + \int_{x_1}^{x_2} \arg \frac{f(u+iy_1)}{f(u+iy_2)} \mathrm{d}u.$$



Basic estimates for $N(\sigma, T)$

Theorem (Bohr–Landau, 1914)

For fixed $\sigma > \frac{1}{2}$, $N(\sigma, T) \ll_\sigma T$.

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Plugging $f = \zeta$ into Littlewood's formula, we have

$$2\pi \int_\sigma^2 N(u, T)\mathrm{d}u = \int_0^T \log|\zeta(\sigma + it)|\mathrm{d}t + O(\log T).$$

By convexity,

$$\begin{aligned}\int_0^T \log|\zeta(\sigma + it)|\mathrm{d}t &= \frac{T}{2} \cdot \frac{1}{T} \int_0^T \log|\zeta(\sigma + it)|^2\mathrm{d}t \\ &\le \frac{T}{2} \log\left\{\frac{1}{T}\int_0^T |\zeta(\sigma + it)|^2\mathrm{d}t\right\}.\end{aligned}$$

Second moment for ζ

For $\sigma > \frac{1}{2}$, it can be shown that

$$\int_0^T |\zeta(\sigma + it)|^2 \mathrm{d}t \ll_\sigma T,$$

so we have

$$\int_\sigma^2 N(u, T)\mathrm{d}u \ll_\sigma T.$$

Set $\sigma_1 = \frac{1}{2} + \frac{1}{2}(\sigma - \frac{1}{2})$. Then by monotonicity,

$$N(\sigma, T) \leq \frac{1}{\sigma - \sigma_1} \int_{\sigma_1}^{\sigma} N(u, T)\mathrm{d}u \ll_\sigma T.$$

Conclusion

$$N(\sigma,T) \ll_\sigma T = o(T\log T) = o(N(T)), \quad \sigma > \frac{1}{2} \text{ fixed.}$$

Theorem (Bohr–Landau, 1914)

Let $\delta > 0$. Then almost all nontrivial zeros $\rho = \beta + i\gamma$ satisfy $|\beta - \frac{1}{2}| < \delta$.

Conclusion

$$N(\sigma,T) \ll_\sigma T = o(T\log T) = o(N(T)), \quad \sigma > \frac{1}{2} \text{ fixed.}$$

Theorem (Bohr–Landau, 1914)

Let $\delta > 0$. Then almost all nontrivial zeros $\rho = \beta + i\gamma$ satisfy $|\beta - \frac{1}{2}| < \delta$.

Remark

Littlewood (1924) improved this to

$$\delta = \frac{\Psi(|\gamma|)\log\log|\gamma|}{\log|\gamma|}$$

provided that $x \to +\infty \Rightarrow \Psi(x) \to +\infty$. Selberg (1942) removed $\log\log|\gamma|$.

Mollifier technique

Bohr and Landau noticed that when $f = \zeta M - 1$, one has

$$h = 1 - f^2 = \zeta M(2 - \zeta M),$$

so a zero of $\zeta(s)$ must also be a zero of h, so by Littlewood's formula,

$$\int_\sigma^2 [N(u, 2T) - N(u, T)]\mathrm{d}u \ll \int_T^{2T} \log|1 - f^2|\mathrm{d}t \ll \int_T^{2T} |f|^2\mathrm{d}t = \int_T^{2T} |\zeta(\sigma + it)M(\sigma + it) - 1|^2\mathrm{d}t,$$

so we can go beyond Littlewood by making $M(s) \approx 1/\zeta(s)$.

$M(s)$ is called the mollifier.

Choice of mollifier

The follow candidates for $M(s)$ have been considered:

- Partial Euler product (Bohr and Landau): $\prod_{p\le X}\left(1-\frac{1}{p^s}\right)$,
- Dirichlet polynomial (Carlson): $\sum_{n\le X}\frac{\mu(n)}{n^s}$,

and $X\to\infty$ as $T\to\infty$.

Technical explanation

When $t \in [T, 2T]$, $\zeta(s)$ can be approximated by a Dirichlet polynomial

$$\zeta(s) = \sum_{n \le T} \frac{1}{n^s} + O(T^{-\sigma}),$$

so under Carlson's choice $M(s) = \sum_{m \le X} \mu(m) m^{-s}$,

$$f(s) \approx \sum_{m \le X} \frac{\mu(m)}{m^s} \sum_{n \le T} \frac{1}{n^s} - 1 = \sum_{X < n \le XT} \frac{a_n}{n^s},$$

where

$$a_n = \sum_{n = ab} \mu(a) \mathbf{1}_{a \le X} \mathbf{1}_{b \le T}.$$

Careful choice of X leads to improved estimates for $N(\sigma, T)$.

Improved estimates for $N(\sigma, T)$

1. $N(\sigma,T) \ll_\varepsilon T^{4\sigma(1-\sigma)+\varepsilon}$.
2. $N(\sigma,T) \ll T^{\frac{3}{2}-\sigma} \log^5 T$, better than 1 in $\sigma \in (\frac{1}{2}, \frac{3}{4})$.
3. If $\zeta(\frac{1}{2}+it) \ll (1+|t|)^\alpha$, then $N(\sigma,T) \ll T^{2(1+2\alpha)(1-\sigma)} \log^5 T$.
4. Ingham (1940): $N(\sigma,T) \ll T^{\frac{3(1-\sigma)}{2-\sigma}} \log^5 T$.

Remark

Since the 1960s, estimates of $N(\sigma,T)$ near $\sigma = 1$ have been improved using "large value" estimates of Dirichlet polynomials, but Ingham's result remained optimal for $\frac{1}{2} \le \sigma \le \frac{3}{4}$ until Guth–Maynard (2024).

Estimates for $N(\sigma, T)$ near $\sigma = \frac{1}{2}$

When $\sigma \to \frac{1}{2}$, the aforementioned results became worse than trivial, and the best estimate in this direction was due to Littlewood (1924):

$$N(\sigma, T) \ll \frac{T}{\sigma - \frac{1}{2}} \log \frac{1}{\sigma - \frac{1}{2}}.$$

By introducing an optimal mollifier, Selberg (1942) removed $\log \frac{1}{\sigma - \frac{1}{2}}$.

Selberg's optimization

To minimize ($s = \frac{1}{2} + it$),

$$\int_T^{T+U} |\zeta(s)M(s) - 1|^2 \mathrm{d}t \ll \int_T^{T+U} |\zeta(s)M(s)|^2 \mathrm{d}t + U,$$

Selberg considered a Dirichlet polynomial with undetermined coefficients:

$$M(s) = \sum_{n \le X} \frac{\lambda_n}{n^s}, \quad \lambda_1 = 1, \lambda_n \in \mathbb{R}$$

so one has

$$\int_T^{T+U} |\zeta(s)M(s)|^2 \mathrm{d}t = \sum_{m,n \le X} \frac{\lambda_m \lambda_n}{\sqrt{mn}} \int_T^{T+U} |\zeta(s)|^2 \left(\frac{n}{m}\right)^{it} \mathrm{d}t.$$

Selberg's optimization

By AFE and saddle point method, Selberg showed that

Lemma

For appropriate X, U, $s = \frac{1}{2} + it$, and $m, n \le X$ coprime, one has

$$\int_T^{T+U} |\zeta(s)|^2 \left(\frac{n}{m}\right)^{it} \mathrm{d}t \sim \frac{U}{\sqrt{mn}} \log \frac{Te^{2\gamma}}{2\pi mn}.$$

Hence, we have

$$\int_T^{T+U} |\zeta(s)M(s)|^2 \mathrm{d}t \sim U \sum_{m,n \le X} \frac{\lambda_m \lambda_n}{mn}(m,n) \log \frac{Te^{2\gamma}(m,n)^2}{2\pi mn}.$$

Selberg's optimization

$$\int_T^{T+U} |\zeta(s)M(s)|^2 \mathrm{d}t \sim U \sum_{m,n \le X} \frac{\lambda_m \lambda_n}{mn} (m,n) \log \frac{Te^{2\gamma}(m,n)^2}{2\pi mn}.$$

The log term is a distraction. It suffices to minimize

$$Q = \sum_{m,n \le X} \frac{\lambda_m \lambda_n}{mn} (m,n).$$

subjected to $\lambda_1 = 1$. By tricky Cauchy–Schwarzing, we arrive at

$$\lambda_n \sim \mu(n) \frac{\log X/n}{\log X}, \quad Q \sim \frac{1}{\log X}.$$

The same optimization procedure was later used in Selberg's Λ^2-sieve (1947).

Conclusion

Eventually, we have for $X = T^{\frac{1}{100}}$ and $U = T^{\frac{14}{15}}$ that

$$\int_T^{T+U} |\zeta(s)M(s)|^2 \mathrm{d}t \ll U\left(\frac{\log T}{\log X}\right) \ll U,$$

which implies

$$\int_{\frac{1}{2}}^{2} [N(u, T+U) - N(u, T)]\mathrm{d}u \ll U,$$

so

$$N(\sigma, T) \leq \frac{1}{\sigma - \frac{1}{2}} \int_{\frac{1}{2}}^{2} N(u, T)\mathrm{d}u \ll \frac{T}{\sigma - \frac{1}{2}}.$$

Conclusion

If $\Psi(x) \to +\infty$, then we see that under $\sigma = \frac{1}{2} + \frac{\Psi(T)}{\log T}$, one has

$$N(\sigma, T) \ll \frac{T \log T}{\Psi(T)} = o(T \log T) = o(N(T)),$$

so

Theorem (Selberg, 1942)

Almost all nontrivial zeros $\rho = \beta + i\gamma$ satsify

$$\left|\beta - \frac{1}{2}\right| < \frac{\Psi(|\gamma|)}{\log|\gamma|}.$$

Moral of Selberg's optimization

Selberg's optimal choice of λ_n satisfies

$$\lambda_n \sim \mu(n)\frac{\log X/n}{\log X} = \mu(n)\left(1 - \frac{\log n}{\log X}\right),$$

which suggests that when making Dirichlet polynomial approximations, it is better to attach a weight:

$$\sum_{n\leq X} \frac{\alpha_n}{n^s}\left(1 - \frac{\log n}{\log X}\right).$$

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Basic zero detecting device

By elementary real analysis, we know that

Proposition

Let f be continuous and real-valued on $[a,b]$. If

$$\left|\int_a^b f(t)\mathrm{d}t\right| < \int_a^b |f(t)|\mathrm{d}t,$$

then f must have a sign change in (a,b).

Task: Find $f(t)$ whose real zeros coincide with those of $\zeta(\frac{1}{2}+it)$.

Hardy's Z-function

Recall from the functional equation that

$$\chi(s) = \frac{\zeta(s)}{\zeta(1-s)} = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s),$$

which has no zeros in $0 < \sigma < 1$, so χ has a well defined square root.

Moreover, $\chi(s)\chi(1-s) = 1$, so when $s = \frac{1}{2} + it$,

$$Z(t) := \chi(s)^{-\frac{1}{2}} \zeta(s)$$

is a real-valued even function satisfying $|Z(t)| = |\zeta(\frac{1}{2} + it)|$.

$N_0(T) \longrightarrow +\infty$

Let

$$I = \int_T^{T+H} Z(t)\mathrm{d}t, \quad J = \int_T^{T+H} |Z(t)|\mathrm{d}t.$$

Then it suffices to show $|I| < J$ for all large T.

By some intensive computations, we have

$$|I| \ll_\varepsilon T^{\frac{3}{4}+\frac{\varepsilon}{2}}, \quad 2 \le H \le T.$$

Estimates for J

For J, one has

$$J = \int_T^{T+H} |\zeta(\tfrac{1}{2} + it)| \mathrm{d}t \geq \left| \int_{\frac{1}{2}+iT}^{\frac{1}{2}+i(T+H)} \zeta(s) \mathrm{d}s \right|.$$

By Cauchy's integral theorem, we can move to $\sigma = 2$,

$$\begin{aligned} \int_{\frac{1}{2}+iT}^{\frac{1}{2}+i(T+H)} \zeta(s) \mathrm{d}s &= i \int_T^{T+H} \sum_{n \geq 1} \frac{1}{n^{2+it}} \mathrm{d}t + O_\varepsilon(T^{\frac{1}{4}+\varepsilon}) \\ &= iH + O\left(\sum_{n \geq 2} \frac{1}{n^2 \log n} \right) + O_\varepsilon(T^{\frac{1}{4}+\varepsilon}), \end{aligned}$$

so for $H \gg T^{\frac{1}{4}+\varepsilon}$, this is $\gg H$.

Conclusion

Setting $H = T^{\frac{3}{4}+\varepsilon}$, we see that

$$J \gg H = T^{\frac{3}{4}+\varepsilon} > T^{\frac{3}{4}+\frac{\varepsilon}{2}} \gg |I|.$$

Theorem (Hardy–Landau)

Let $\varepsilon > 0$. Then for all $T \geq T_0(\varepsilon)$, $Z(t)$ has a sign change in $(T, T + T^{\frac{3}{4}+\varepsilon})$.

Exercise: Deduce that $N_0(T) \gg T^{\frac{1}{4}-\varepsilon}$.

Hardy–Littlewood zero detecting device

Instead of considering one interval, we look at multiple intervals together:

$$I(t) = \int_t^{t+H} f(u)\mathrm{d}u, \quad J(t) = \int_t^{t+H} |f(u)|\mathrm{d}u,$$

where $f(u)$ is related to $\zeta(\frac{1}{2} + iu)$ and $t \asymp T$.

Key observation: If f has no zeros in $(t_0, t_0 + 2H)$, then $|I(t)| = J(t)$ in $(t_0, t_0 + H)$.

Hardy–Littlewood zero detecting device

Define

$$S = \{t \in (T, 2T) : |I(t)| = J(t)\}$$

and a system of pairwise disjoint intervals:

$$j_k = (T + 2(k-1)H, T + 2kH), \quad 1 \le k \le m = \lfloor T/2H \rfloor.$$

If f does not change sign in $j_{k_1}, j_{k_2}, \dots, j_{k_r}$, then

$$rH \le \sum_{\ell=1}^{r} \mu(j_{k_\ell} \cap S) \le \mu(S) \Rightarrow r \le \frac{\mu(S)}{H},$$

where μ is the Lebesgue measure on $\mathbb{R}$, so

$$N_0(2T) - N_0(T) \ge m - r \ge \frac{T}{2H} - \frac{\mu(S)}{H} - 1.$$

Problem setup

Since

$$N_0(2T) - N_0(T) \geq m - r \geq \frac{T}{2H} - \frac{\mu(S)}{H} - 1, \tag{1}$$

it suffices to provide an upper bound for $\mu(S)$. Since

$$\int_S J(t)\mathrm{d}t = \int_S |I(t)|\mathrm{d}t \leq \mu(S)^{\frac{1}{2}} \|I\|_{L^2}, \tag{2}$$

so we need an upper bound for $\|I\|_{L^2}$ and a lower bound for $J(t)$.

Upper bound for $\|I\|_{L^2}$

We choose

$$f(u) = Z(u) = \chi(\tfrac{1}{2} + iu)^{-\frac{1}{2}}\zeta(\tfrac{1}{2} + iu).$$

After some brute-force computations, one has $\|I\|_{L^2} \ll H^{\frac{1}{2}}T^{\frac{1}{2}}$, so the inequality (2) becomes

$$\mu(S)^{\frac{1}{2}}H^{\frac{1}{2}}T^{\frac{1}{2}} \gg \int_S J(t)\mathrm{d}t.$$

Lower bound for $J(t)$

Since $|Z(u)| = |\zeta(\frac{1}{2} + iu)|$, one has

$$J(t) \geq \left| \int_t^{t+H} \zeta(\tfrac{1}{2} + iu) \mathrm{d}u \right|.$$

Using the approximation $\zeta(\frac{1}{2} + iu) = \sum_{n \leq T} n^{-\frac{1}{2} - iu} + O(T^{-\frac{1}{2}})$, one has

$$\int_t^{t+H} \zeta(\tfrac{1}{2} + iu) \mathrm{d}u = H - iG(t) + O(HT^{-\frac{1}{2}}),$$

where

$$G(t) = \sum_{2 \leq n \leq T} \frac{1 - n^{-iH}}{n^{\frac{1}{2} + it} \log n}.$$

Lower bound for $J(t)$

Thus, we have

$$J(t) > C_1 H - C_2 |G(t)|.$$

Integrating over S, we get

$$\mu(S)^{\frac{1}{2}} H^{\frac{1}{2}} T^{\frac{1}{2}} > C_1 \mu(S) H - C_2 \int_S |G(t)| \mathrm{d}t.$$

By Cauchy–Schwarz and Parseval, one has

$$\int_S |G(t)| \mathrm{d}t \ll \mu(S)^{\frac{1}{2}} T^{\frac{1}{2}},$$

so $\mu(S)H \ll \mu(S)^{\frac{1}{2}} H^{\frac{1}{2}} T^{\frac{1}{2}} + \mu(S)^{\frac{1}{2}} T^{\frac{1}{2}}$, meaning for $H \geq 1$,

$$\mu(S) < C_3 T H^{-1}.$$

Conclusion

Plugging $\mu(S) < C_3 T H^{-1}$ into the lower bound (1), we have

$$N_0(2T) - N_0(T) > \frac{T}{2H} - \frac{C_3 T}{H^2} = \frac{T}{H}\left(\frac{1}{2} - \frac{C_3}{H}\right),$$

which is $\gg TH^{-1}$ provided that H is large and fixed. Hence, we have

Theorem (Hardy–Littlewood, 1921)

As $T \to +\infty$, $N_0(T) \gg T$.

The Hardy–Littlewood framework

The proof method is summarized as follows

- Let $H > 0$ be an undetermined parameter.
- Estimate $I(t) = \int_t^{t+H} f(u)\mathrm{d}u$ and $J(t) = \int_t^{t+H} |f(u)|\mathrm{d}u$.
- Bound the measure of $S = \{t \in (T, 2T) : |I(t)| = J(t)\}$.
- Plugging the upper bound estimate for $\mu(S)$ into

$$N_0(2T) - N_0(T) \geq \frac{T}{2H} - \frac{\mu(S)}{H} - 1$$

and conclude.

Problem: HL's choice of f does not allow this argument to run when $H \to 0$.

Hardy–Littlewood framework with mollifiers

Selberg (1942) overcame the problem by attaching a mollifier:

$$f(u) = Z(u)|\phi(\tfrac{1}{2} + iu)|^2,$$

where $\phi(s)$ is an approximation to $\{\zeta(s)\}^{-\frac{1}{2}}$.

Slogan: It is easier to work with even powers of modulus.

Selberg's choice of mollifier

Based on our experience with $N(\sigma, T)$, the optimal $\phi(s)$ takes the form of

$$\phi(s) = \sum_{n\le X} \frac{\alpha_n}{n^s}\left(1 - \frac{\log n}{\log X}\right),$$

where α_n satisfies $\sum_{n\ge1} \alpha_n n^{-s} = \{\zeta(s)\}^{-\frac{1}{2}}$.

This allows Selberg to obtain $\mu(S) < C_4 T/\sqrt{H\log T}$, so

$$N_0(2T) - N_0(T) > \frac{T}{H}\left(\frac{1}{2} - \frac{C_4}{\sqrt{H\log T}}\right),$$

which is $\gg T\log T$ provided that $H = A/\log T$ and $A \gg 1$.

Remark on Selberg's method

Before obtaining $N_0(T) \gg T\log T$, Selberg also worked with partial Euler product

$$\phi(s) = \prod_{p\le X}\left(1 - \frac{1}{2p^s} - \frac{1}{8p^{2s}}\right)$$

but only achieved $N_0(T) \gg T\log\log\log T$.

Note: By binomial theorem,

$$\left(1 - \frac{1}{p^s}\right)^{\frac{1}{2}} = 1 - \frac{1}{2p^s} - \frac{1}{8p^{2s}} + \dots$$

Further works

- Selberg (1942):

$$\alpha = \liminf_{T\to+\infty} \frac{N_0(T)}{N(T)} > 0.$$

After Selberg, there are further developments on α:

- S. H. Min 闵嗣鹤 (1956): $\alpha \geq \frac{1}{60000}$ by tracking Selberg's constants,
- Levinson (1974): $\alpha \geq 0.342$ using a very different method,
- Conrey (1989): $\alpha \geq 0.4077$ via Kloosterman sums, etc.;
- Pratt–Robles–Zaharescu–Zeindler (2020): $\alpha \geq 0.416$.

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