Special values of the Riemann zeta function

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Introduction

In 1734, **Leonard Euler** solved the Basel problem by showing that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

This was subsequently generalized to

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k},$$

where B_m is the m'th **Bernoulli number** satisfying

$$B_0 = 1, \quad B_k = -\frac{1}{k+1} \sum_{m=0}^{k-1} \binom{k+1}{m} B_m.$$

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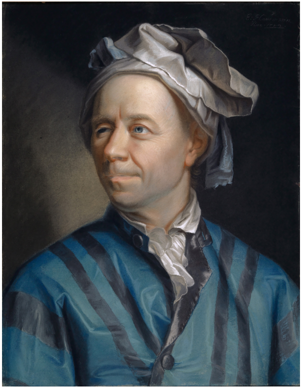
Euler's solution



Leonard Euler

Karl Weierstrass

Euler's solution

Define

$$F(x) = \frac{\sin x}{x}, \quad F_N(x) = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\cdots\left(1 - \frac{x^2}{N^2\pi^2}\right).$$

Euler claimed that $F_N(x) \to F(x)$ as $x \to +\infty$.

$$F_N(x) = 1 - \left(\sum_{n=1}^{N} \frac{1}{n^2\pi^2}\right) x^2 + \text{higher order terms}$$

Therefore, the coefficient of x^2 in $F(x)$ should be

$$\lim_{N\to+\infty} -\left(\sum_{n=1}^{N} \frac{1}{n^2\pi^2}\right) = -\frac{\zeta(2)}{\pi^2}.$$

Euler's solution

By Taylor's expansion, we know

$$\sin x = x - \frac{x^3}{6} + \text{higher order terms},$$

so

$$F(x) = \frac{\sin x}{x} = x - \frac{x^2}{6} + \text{higher order terms},$$

which means

$$-\frac{\zeta(2)}{\pi^2} = -\frac{1}{6} \Rightarrow \zeta(2) = \frac{\pi^2}{6}.$$

Remark on Euler's solution

Euler took $F_N(x) \to F(x)$ for granted. The first rigorous proof was due to **Karl Weierstrass** via complex analysis:

Theorem (Weierstrass)

Let $F : \mathbb{C} \to \mathbb{C}$ be analytic with zeros $a_1, a_2, \ldots$ counted with multiplicity and $F(0) \neq 0$. If

$$\frac{1}{|a_1|^2} + \frac{1}{|a_2|^2} + \cdots < \infty$$

then there is some analytic $g : \mathbb{C} \to \mathbb{C}$ such that

$$F(z) = e^{G(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n}.$$

Rigorous proof

Observe that $F(z) = (\sin z)/z$ is analytic on $\mathbb{C}$ and has zeros only at

$$z = \pi, -\pi, 2\pi, -2\pi, 3\pi, -3\pi, \ldots$$

and

$$\frac{1}{(\pi)^2} + \frac{1}{(-\pi)^2} + \frac{1}{(2\pi)^2} + \frac{1}{(-2\pi)^2} + \cdots < \infty,$$

so we use Weierstrass's theorem to conclude that

$$\begin{aligned} F(z) &= e^{G(z)} \prod_{m=1}^{\infty} \left(1 - \frac{z}{m\pi}\right) e^{z/m} \prod_{n=1}^{\infty} \left(1 - \frac{z}{-n\pi}\right) e^{-z/n} \\ &= e^{G(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) := e^{G(z)} H(z). \end{aligned}$$

Rigorous proof

$$H(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right), \quad F(z) = e^{G(z)}H(z).$$

Since $H(0) = F(0) = 1$, we take $G(0) = 0$. It can be shown that $|F(z)/H(z)| \le \exp(|z|^{1.01})$ as $|z| \to +\infty$, so $\Re[G(z)] \le |z|^{1.01}$.

Theorem (Borel–Carathéodory)

Let $G : \mathbb{C} \to \mathbb{C}$ be analytic with $G(0) = 0$. If $\Re[G(z)] \le |z|^s$ for some $s > 0$, then $G(z)$ is a polynomial of degree $\lfloor s \rfloor$.

By this theorem, $G(z) = Az$ for some $A \in \mathbb{C}$. Because $F(z)/H(z)$ is even, we conclude $A = 0$, so

$$F(x) = \frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right) = H(x).$$

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Sum of powers

Define $S_p(n)$ as

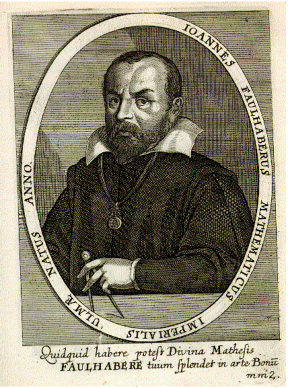
$$S_p(n) = 1^p + 2^p + \cdots + n^p = \sum_{k=1}^{n} k^p.$$

Then

$$S_0(n) = n, \quad S_1(n) = \frac{n(n+1)}{2}, \quad S_2(n) = \frac{n(n+1)(2n+1)}{6}.$$

Generalizations?

Faulhaber's formula



Johann Faulhaber

Jacob Bernoulli

Faulhaber's formula

In the 17th century, **Johann Faulhaber** derived formulas for $S_p(n)$ for $p \leq 17$. In 1713, **Jacob Bernoulli** introduced Bernoulli numbers B_m and showed that

Theorem (Bernoulli)

For $p, n \in \mathbb{N}$, there is

$$S_p(n) = \frac{1}{p+1} \sum_{m=0}^{p} \binom{p+1}{m} (-1)^m B_m n^{p+1-m}.$$

Proof of Faulhaber's formula

The most efficient proof is through generating functions:

$$E(x) = \sum_{p=0}^{\infty} \frac{S_p(n)}{p!} x^p.$$

By interchanging order of summation,

$$E(x) = \sum_{k=1}^{n} \sum_{p=0}^{\infty} \frac{(kx)^p}{p!} = \sum_{k=1}^{n} e^{kx} = e^x \cdot \frac{e^{nx}-1}{e^x-1}$$
$$= \frac{e^{nx}-1}{x} \cdot \frac{-x}{e^{-x}-1}.$$

Proof of Faulhaber's formula

Definition

Define m'th **Bernoulli number** B_m as the coefficient of

$$\frac{y}{e^y - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} y^m.$$

Thus, we have

$$\frac{e^{nx} - 1}{x} \cdot \frac{-x}{e^{-x} - 1} = \sum_{k=0}^{\infty} \frac{n^{k+1} x^k}{(k+1)!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} B_m x^m.$$

By comparing coefficients,

$$\frac{S_p(n)}{p!} = \sum_{\substack{m,k \geq 0 \\ m+k=p}} \frac{(-1)^m B_m ! n^{k+1}}{m!(p+1-m)!}.$$

Proof of Faulhaber's formula

By comparing coefficients,

$$\frac{S_p(n)}{p!} = \sum_{\substack{m,k\geq 0 \\ m+k=p}} \frac{(-1)^m B_m! n^{k+1}}{m!(k+1)!}.$$

Rearranging gives

$$\begin{aligned} S_p(n) &= \sum_{m=0}^{p} \frac{p!}{m!(p+1-m)!} (-1)^m B_m n^{p+1-m} \\ &= \frac{1}{p+1} \sum_{m=0}^{p} \binom{p+1}{m} (-1)^m B_m n^{p+1-m}. \end{aligned}$$

How to compute B_m?

Computation of Bernoulli numbers

Note that

$$1 = \frac{e^y - 1}{y} \cdot \frac{y}{e^y - 1},$$

so we have

$$1 = \sum_{k=0}^{\infty} \frac{y^k}{(k+1)!} \sum_{m=0}^{\infty} \frac{B_m}{m!} y^m.$$

Comparing coefficients gives

$$\sum_{m=0}^{p} \binom{p+1}{m} B_m = \begin{cases} 1 & p = 0, \\ 0 & p > 0. \end{cases}$$

Recursive formula for B_m

$$\sum_{m=0}^{p} \binom{p+1}{m} B_m = \begin{cases} 1 & p = 0, \\ 0 & p > 0. \end{cases}$$

Let $p = 0$, so $B_0 = 1$. For $p > 0$

$$(p+1)B_p + \sum_{m=0}^{p-1} \binom{p+1}{m} B_m = 0,$$

so we have for $p > 0$ that

$$B_p = \frac{-1}{p+1} \sum_{m=0}^{p-1} \binom{p+1}{m} B_m.$$

Computation gives $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, and $B_4 = -\frac{1}{30}$.

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Formula for $\zeta(2k)$

In this section, we prove the formula for $\zeta(2k)$ by expressing $\cot x$ in two ways:

$$\cot x = \frac{1}{x} - 2\sum_{k=1}^{\infty} \frac{x^{2k-1}}{\pi^{2k}} \zeta(2k) = i + \frac{2i}{e^{2ix} - 1}.$$

Then, the result is proved by comparing coefficients with

$$\frac{y}{e^y - 1} = \sum_{m \geq 0} \frac{B_m}{m!} y^m.$$

Expression of $\cot x$ via $\zeta(2k)$

Taking logarithms on both sides of

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right)$$

gives

$$\log\sin x = \log x + \sum_{n=1}^{\infty}\log\frac{n^2\pi^2 - x^2}{n^2\pi^2}.$$

Differentiating gives

$$\cot x = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{-\frac{2x}{n^2\pi^2}}{1 - \frac{x^2}{n^2\pi^2}}.$$

Expression of $\cot x$ via $\zeta(2k)$

$$\cot x = \frac{1}{x} - \frac{2}{x}\sum_{n=1}^{\infty} \frac{\frac{x^2}{n^2\pi^2}}{1 - \frac{x^2}{n^2\pi^2}}.$$

From the geometric series formula,

$$\frac{\frac{x^2}{n^2\pi^2}}{1 - \frac{x^2}{n^2\pi^2}} = \sum_{k=1}^{\infty} \frac{x^{2k}}{n^{2k}\pi^{2k}},$$

so there is

$$\cot x = \frac{1}{x} - \frac{2}{x}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{x^{2k}}{n^{2k}\pi^{2k}} = \frac{1}{x} - 2\sum_{k=1}^{\infty} \frac{x^{2k-1}}{\pi^{2k}} \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{2k}}}_{\zeta(2k)}.$$

Expression of $\cot x$ via Bernoulli numbers

By Euler's identities,

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

we can rewrite $\cot x$ into

$$\begin{aligned} \cot x &= i \cdot \frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}} = i \cdot \frac{e^{2ix} + 1}{e^{2ix} - 1} \\ &= i \cdot \frac{e^{2ix} - 1 + 2}{e^{2ix} - 1} = i + \frac{2i}{e^{2ix} - 1}, \end{aligned}$$

so there is

$$\cot x = i + \frac{1}{x} \cdot \frac{2ix}{e^{2ix} - 1} = i + \frac{1}{x} \sum_{m=0}^{\infty} \frac{B_m}{m!} (2ix)^m.$$

Completion of proof

Equating two expressions for $\cot x$ gives

$$\frac{1}{x} - 2\sum_{k=1}^{\infty} \frac{x^{2k-1}}{\pi^{2k}}\zeta(2k) = i + \frac{1}{x}\cdot\frac{2ix}{e^{2ix}-1} = i + \frac{1}{x}\sum_{m=0}^{\infty}\frac{B_m}{m!}(2ix)^m,$$

so by matching the coefficients of x^{2k-1}, we have

$$-\frac{2\zeta(2k)}{\pi^{2k}} = \frac{B_{2k}}{(2k)!}(2i)^{2k} = (-1)^k\frac{2^{2k}B_{2k}}{(2k)!}.$$

$$\zeta(2k) = (-1)^{k+1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}.$$

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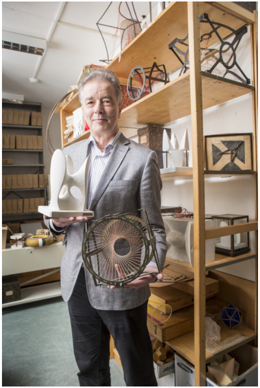
Irrationality of $\zeta(3)$

Roger Apéry



Frits Beukers

Apéry's Theorem

In June 1978, **Roger Apéry** spoke at Journées Arithmétiques de Marseille-Luminy to present his proof of

Theorem (Apéry, 1977)

$\zeta(3)$ is irrational.

Apéry's proof was based on a carefully constructed series that converges to $\zeta(3)$ rapidly. In 1979, **Frits Beukers** gave a simpler proof based on a carefully designed double integral.

Irrationality and rational approximation

If $x = \frac{h}{k} \in \mathbb{Q}$, then for any $\frac{p}{q} \in \mathbb{Q} \setminus \{x\}$, there is

$$\left| x - \frac{p}{q} \right| = \frac{|hq - pk|}{kq} \geq \frac{1}{kq}.$$

This provides the following criterion for irrationality:

Theorem (Dirichlet)

Let $x \in \mathbb{R}$. If there exists some $\delta > 0$ such that there are infinitely many $\frac{p}{q} \in \mathbb{Q}$ such that p, q are coprime and

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{1+\delta}},$$

then x is irrational.

Outline of proof

We use Beukers's method for simplicity.

- Step 1: Integral for $\zeta(3)$: for $f \in \mathbb{Z}[x]$ of degree n,

$$I(f) = \int_0^1 \int_0^1 \frac{\log(uv)^{-1}}{1 - uv} f(u) f(v) \mathrm{d}u \mathrm{d}v = \frac{A\zeta(3) - B}{D_n^3}$$

  for $A \in \mathbb{Z} \setminus \{0\}$, $B \in \mathbb{Z}$, and $D_n = \mathrm{lcm}(1, 2, \ldots, n)$.

- Step 2: If $\zeta(3) = \frac{h}{k}$, then

$$1 \leq |Ah - Bk| = kD_n^3 |I(f)|.$$

- Step 3: Find some $f \in \mathbb{Z}[x]$ such that $kD_n^3 |I(f)| < 1$, so there is a contradiction.

Step 1: $I_{r,s}$ and $J_{m,n}$

It suffices to study

$$I_{r,s} = \int_0^1 \int_0^1 \frac{\log(uv)^{-1}}{1-uv} u^r v^s \mathrm{d}u\mathrm{d}v.$$

From the formula of geometric series, we have

$$\frac{1}{1-z} = \sum_{k\geq 1} z^{k-1} \Rightarrow I_{r,s} = \sum_{k\geq 1} J_{k+r,k+s},$$

where

$$J_{m,n} = \int_0^1 \int_0^1 u^{m-1} v^{n-1} \log(uv)^{-1} \mathrm{d}u\mathrm{d}v.$$

Step 1: Evaluation of $J_{m,n}$

By integration by parts, there is

$$\int_0^1 \log(u)^{-1} u^{m-1} \mathrm{d}u = \frac{1}{m^2}$$

$$\Rightarrow \int_0^1 u^{m-1} v^{n-1} \log(uv)^{-1} \mathrm{d}u = \frac{v^{n-1}}{m^2} + \frac{v^{n-1}}{m} \log(v)^{-1}$$

$$\Rightarrow J_{m,n} = \frac{1}{m^2 n} + \frac{1}{n^2 m}.$$

Step 1: Evaluation of $I_{r,s}$

$$I_{r,s} = \sum_{k\geq 1} J_{k+r,k+s} = \sum_{n>r} J_{n,n+s-r}.$$

If $r = s$, then

$$I_{r,r} = \sum_{n>r} J_{n,n} = 2\sum_{n>r}\frac{1}{n^3} = 2\left[\zeta(3) - \sum_{n=1}^{r}\frac{1}{n^3}\right].$$

If $r \neq s$, it follows from $I_{r,s} = I_{s,r}$ that we assume WLOG $r < s$:

$$\begin{aligned} I_{r,s} &= \sum_{n>r}\frac{1}{n(n+s-r)}\left[\frac{1}{n}+\frac{1}{n+s-r}\right] \\ &= \frac{1}{s-r}\sum_{n>r}\left[\frac{1}{n}-\frac{1}{n+s-r}\right]\left[\frac{1}{n}+\frac{1}{n+s-r}\right] \end{aligned}$$

Step 1: Evaluation of $I_{r,s}$ (continued)

$$I_{r,s} = \frac{1}{s-r} \sum_{n>r} \left[\frac{1}{n^2} - \frac{1}{(n+s-r)^2} \right] = \frac{1}{s-r} \sum_{r<n\le s} \frac{1}{n^2}.$$

Since $(s-r)n^2 \le s^3$, $D_s^3 I_{r,s} \in \mathbb{Z}$, where $D_n = \operatorname{lcm}(1,2,\ldots,n)$. Conclusively, when $0 \le r, s \le n$, there is

$$I_{r,s} = \begin{cases} 2\left[\zeta(3) - \sum_{m=1}^{r} \frac{1}{m^3}\right] & r = s \\ K_{r,s}/D_n^3 & r \ne s \quad (K_{r,s} \in \mathbb{Z}). \end{cases}$$

Step 1: Evaluation of $I(f)$

$$I_{r,s} = \begin{cases} 2\left[\zeta(3) - \sum_{m=1}^{r} \frac{1}{m^3}\right] & r = s \\ K_{r,s}/D_n^3 & r \neq s \quad (K_{r,s} \in \mathbb{Z}). \end{cases}$$

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ for $a_0, a_1, \ldots, a_n \in \mathbb{Z}$, then

$$\begin{aligned} I(f) &= \sum_{0 \le r,s \le n} a_r a_s I_{r,s} = \sum_{r=0}^{n} a_r^2 I_{r,r} + \sum_{\substack{0 \le r,s \le n \\ r \neq s}} a_r a_s I_{r,s} \\ &= 2\sum_{r=0}^{n} a_r^2 \zeta(3) - 2\sum_{r=0}^{n} a_r^2 \sum_{m=1}^{r} \frac{1}{m^3} + \sum_{\substack{0 \le r,s \le n \\ r \neq s}} \frac{a_r a_s K_{r,s}}{D_n^3} \\ &= \frac{A\zeta(3) - B}{D_n^3} \quad (A \in \mathbb{Z} \setminus \{0\}, B \in \mathbb{Z}). \end{aligned}$$

Step 3: Bounds for $I(f)$

Recall that

$$I(f) = \int_0^1 \int_0^1 \frac{\log(uv)^{-1}}{1-uv} f(u) f(v) \mathrm{d}u \mathrm{d}v,$$

so it follows from

$$\log \alpha^{-1} = \int_\alpha^1 \frac{\mathrm{d}v}{v} = \int_0^{1-\alpha} \frac{\mathrm{d}v}{1-v} = (1-\alpha) \int_0^1 \frac{\mathrm{d}z}{1-(1-\alpha)z}$$

that

$$I(f) = \int_0^1 \int_0^1 \int_0^1 \frac{f(u) f(v)}{1-(1-uv)z} \mathrm{d}u \mathrm{d}v \mathrm{d}z.$$

Let f be defined as

$$f(x) = \frac{1}{n!} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^n [x^n (1-x)^n].$$

Step 3: Bounds for $I(f)$

By repeated integration by parts, there is

$$\int_0^1 \int_0^1 \frac{f(u)}{1-(1-uv)z} \mathrm{d}u = \int_0^1 \frac{u^n v^n z^n (1-u)^n}{[1-(1-uv)z]^{n+1}} \mathrm{d}u. \qquad (*)$$

$$\Rightarrow I(f) = \int_0^1 \int_0^1 \int_0^1 \frac{u^n v^n z^n (1-u)^n f(v)}{[1-(1-uv)z]^{n+1}} \mathrm{d}u\mathrm{d}v\mathrm{d}z.$$

Let $z = (1-w)[1-(1-uv)w]^{-1}$. Then

$$I(f) = \int_0^1 \int_0^1 \int_0^1 (1-u)^n (1-w)^n \frac{f(v)}{1-(1-uv)w} \mathrm{d}u\mathrm{d}v\mathrm{d}w$$

Applying $(*)$ once again, there is

Step 3: Bounds for $I(f)$

$$I(f) = \int_0^1 \int_0^1 \int_0^1 \frac{[u(1-u)v(1-v)w(1-w)]^n}{[1-(1-uv)w]^{n+1}} \mathrm{d}u\mathrm{d}v\mathrm{d}w.$$

Since $1-(1-uv)w = (1-w) + uvw \geq 2\sqrt{1-w}\sqrt{uvw}$

$$Q = \frac{u(1-u)v(1-v)w(1-w)}{1-(1-uv)w} \leq \frac{1}{2} u^{\frac{1}{2}}(1-u)v^{\frac{1}{2}}(1-v)[w(1-w)]^{\frac{1}{2}}.$$

By differential calculus, there is

$$\sup_{u\in[0,1]} u^{\frac{1}{2}}(1-u) = \frac{2}{3\sqrt{3}}, \qquad \sup_{w\in[0,1]} w(1-w) = \frac{1}{4},$$

so we have $Q \leq \frac{1}{4}\left(\frac{2}{3\sqrt{3}}\right)^2 = \frac{1}{27}$.

Step 3: Bounds for $I(f)$

$$|I(f)| \le \left(\frac{1}{27}\right)^n \int_0^1 \int_0^1 \int_0^1 \frac{\mathrm{d}u\mathrm{d}v\mathrm{d}w}{1-(1-uv)w} = C\left(\frac{1}{27}\right)^n.$$

Let $p_1, p_2, \ldots, p_{k_n}$ be all primes $\le n$. Then

$$D_n = p_1^{\lfloor \log_{p_1} n \rfloor} p_2^{\lfloor \log_{p_2} n \rfloor} \cdots p_{k_n}^{\lfloor \log_{p_{k_n}} n \rfloor} \le e^{k_n \log n}.$$

By the prime number theorem, there is

$$\lim_{n \to +\infty} \frac{k_n}{n/\log n} = 1.$$

This means for all $\varepsilon > 0$, $k_n < (1+\varepsilon)n/\log n$ for $n \ge n_0(\varepsilon)$, so $D_n < e^{(1+\varepsilon)n}$

Finishing step

$$D_n^3|I(f)| < Ce^{3n(1+\varepsilon)}\left(\frac{1}{27}\right)^n = C\left(\frac{e^{1+\varepsilon}}{3}\right)^{3n}.$$

Choose $\varepsilon > 0$ small so that $e^{1+\varepsilon} < 3$. This ensures $D_n^3|I(f)|$ can be made arbitrarily small (e.g $< 1/k$).

If $\zeta(3) = \frac{h}{k}$, then

$$1 \leq |Ah - Bk| = kD_n^3|I(f)| < 1,$$

which is a contradiction.

Bibliography

Tom M. Apostol (1976).
Introduction to Analytic Number Theory.
Springer.

F. Beukers (1979).
A Note on the Irrationality of $\zeta(2)$ and $\zeta(3)$.
Bull. London Math. Soc., **11** (3), 268–272

Kenneth Ireland , Michael Rosen (1990).
A Classical Introduction to Modern Number Theory.
Springer.

Alfred van der Poorten (1979).
A Proof that Euler Missed ...
https://pracownicy.uksw.edu.pl/mwolf/Poorten_MI_195_0.pdf

E. C. Titchmarsh (1951).
The Theory of the Riemann Zeta-Function.
Oxford University Press.