# Special values of the Riemann zeta function

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# Table of Contents

# Table of Contents

# Leonhard Euler (1707-1783)

The zeta function was first introduced by **Leonhard Euler**.

In particular, he proved $\sum\limits_{\text{p prime}} \frac{1}{p}$ diverges.

## Euler product formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

# Bernhard Riemann (1826-1866)

**Bernhard Riemann** extended the Euler definition to a complex variable.

## Definition

The **Riemann zeta function**, denoted by the Greek letter $\zeta$ (zeta), is a mathematical function of a complex variable defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $\Re(s) > 1$, and its analytic continuation everywhere.

# Table of Contents

# Bernoulli Numbers

## Theorem

$$\zeta(2m) = (-1)^{m+1}\frac{(2\pi)^{2m}}{2(2m)!}B_{2m}$$

## Generating function

$$f(x) = \frac{x}{e^x - 1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}x^n \qquad (1)$$

# Bernoulli Numbers

$$B_0 = 1$$
$$B_1 = -\frac{1}{2}$$
$$B_2 = \frac{1}{6}$$
$$B_3 = 0$$
$$B_4 = -\frac{1}{30}$$
$$B_5 = 0$$
$$B_6 = \frac{1}{42}$$
$$B_7 = 0$$
$$B_8 = -\frac{1}{30}$$

## Proposition

For $k > 1$ and odd, $B_k = 0$.

## Proof

$g(x) = \frac{x}{e^x - 1} + \frac{x}{2} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n + \frac{x}{2}$ is even function

$\Rightarrow$ coefficients of $x^m = 0$ where $m$ is odd

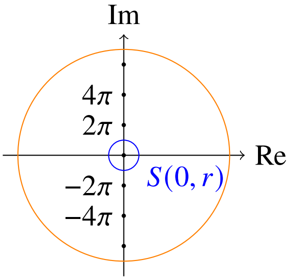
$\Rightarrow$

$$\text{For } m > 1 \text{ and odd, } B_m = 0.$$

$$\text{For } m = 1,\ B_1 x + \frac{x}{2} = 0,\ B_1 = -\frac{1}{2}.$$

# Table of Contents

# Zeta function at even integers




$$C_N : |z| = 2\pi\left(N + \frac{1}{2}\right)$$

By Cauchy's integral formula,

$$\frac{B_n}{n!} = \frac{1}{2\pi i}\int_{S(0,r)} \frac{f(z)}{z^{n+1}}\,dz.$$

Since $f(z)$ has simple poles at $z = 2\pi i k$,

$$\frac{B_n}{n!} = -\sum_{\substack{k=-N \\ k\neq 0}}^{N} \operatorname*{Res}_{z=2\pi i k} \frac{f(z)}{z^{n+1}} + R_N,$$

where

$$R_N = \frac{1}{2\pi i}\int_{C_N} \frac{f(z)}{z^{n+1}}\,dz.$$

# Evaluation of residues

$$\begin{aligned}\operatorname*{Res}_{z=2\pi ik}\frac{f(z)}{z^{n+1}} &= \lim_{z\to 2\pi ik}(z-2\pi ik)\frac{f(z)}{z^{n+1}}\\ &= \lim_{z\to 2\pi ik}\left(\frac{z-2\pi ik}{e^z-1}\right)\left(\frac{1}{z^n}\right)=\frac{1}{(2\pi ik)^n}\end{aligned}$$

$$\Rightarrow \frac{B_n}{n!} = -\sum_{\substack{k=-N\\k\neq 0}}^{N}\frac{1}{(2\pi ik)^n}+R_N = -\sum_{k=1}^{N}\left(\frac{1}{(2\pi ik)^n}+\frac{(-1)^n}{(2\pi ik)^n}\right)+R_N$$

$$\Rightarrow \frac{B_n}{n!} = -\frac{1+(-1)^n}{(2\pi i)^n}\sum_{k=1}^{N}\frac{1}{k^n}+R_N \tag{2}$$

# Bounds for $R_N$

There exists some $M > 0$ such that $|e^z - 1| \geq 1/M$ for all $N \in \mathbb{N}$ and all $z \in C_N$, so when $R = 2\pi\left(N + \frac{1}{2}\right)$, there is

$$|R_N| \leq \frac{1}{2\pi} \sup_{z \in C_N} \left|\frac{f(z)}{z^{n+1}}\right| \cdot 2\pi R \leq \frac{M}{R^n} \cdot R$$

$$= \frac{M}{(2\pi)^{n-1}\left(N + \frac{1}{2}\right)^{n-1}} \to 0 \quad (N \to +\infty, n > 1)$$

# Final result

$$\frac{B_n}{n!} = -\frac{1+(-1)^n}{(2\pi i)^n}\sum_{k=1}^{\infty}\frac{1}{k^n}$$

$$\frac{B_{2m}}{(2m)!} = -\frac{2}{(2\pi i)^{2m}}\sum_{k=1}^{\infty}\frac{1}{k^{2m}} = (-1)^{m+1}\cdot\frac{2}{(2\pi)^{2m}}\zeta(2m)$$

Conclusively, there is

$$\zeta(2m) = (-1)^{m+1}\frac{(2\pi)^{2m}}{2(2m)!}B_{2m}.$$

# Table of Contents

# Apéry's Theorem

In June 1978, **Roger Apéry** spoke at Journées Arithmétiques de Marseille-Luminy to present his proof of

**Theorem (Apéry, 1977)**

*$\zeta(3)$ is irrational.*

In 1979, **Frits Beukers** gave a simpler proof based on integrals.

# Irrationality and rational approximation

If $x = \frac{h}{k} \in \mathbb{Q}$, then for any $\frac{p}{q} \in \mathbb{Q} \setminus \{x\}$, there is

$$\left| x - \frac{p}{q} \right| = \frac{|hq - pk|}{kq} \geq \frac{1}{kq}.$$

This provides the following criterion for irrationality:

## Theorem (Dirichlet)

*Let $x \in \mathbb{R}$. If there exists some $\delta > 0$ such that there are infinitely many $\frac{p}{q} \in \mathbb{Q}$ such that $p, q$ are coprime and*

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{1+\delta}},$$

*then $x$ is irrational.*

# Outline of proof

We use Beukers's method for simplicity.

- Step 1: Integral for $\zeta(3)$: for $f \in \mathbb{Z}[x]$ of degree $n$,

$$I(f) = \int_0^1 \int_0^1 \frac{\log(uv)^{-1}}{1-uv} f(u)f(v)\mathrm{d}u\mathrm{d}v = \frac{A\zeta(3) - B}{D_n^3}$$

  for $A \in \mathbb{Z} \setminus \{0\}$, $B \in \mathbb{Z}$, and $D_n = \mathrm{lcm}(1, 2, \ldots, n)$.
- Step 2: If $\zeta(3) = \frac{h}{k}$, then

$$1 \leq |Ah - Bk| = kD_n^3 |I(f)|.$$

- Step 3: Find some $f \in \mathbb{Z}[x]$ such that $kD_n^3 |I(f)| < 1$, so there is a contradiction.

# Step 1: $I_{r,s}$ and $J_{m,n}$

It suffices to study

$$I_{r,s} = \int_0^1 \int_0^1 \frac{\log(uv)^{-1}}{1-uv} u^r v^s \mathrm{d}u\mathrm{d}v.$$

From the formula of geometric series, we have

$$\frac{1}{1-z} = \sum_{k\geq 1} z^{k-1} \Rightarrow I_{r,s} = \sum_{k\geq 1} J_{k+r,k+s},$$

where

$$J_{m,n} = \int_0^1 \int_0^1 u^{m-1} v^{n-1} \log(uv)^{-1} \mathrm{d}u\mathrm{d}v.$$

# Step 1: Evaluation of $J_{m,n}$

By integration by parts, there is

$$\int_0^1 \log(u)^{-1} u^{m-1} \mathrm{d}u = \frac{1}{m^2}$$

$$\Rightarrow \int_0^1 u^{m-1} v^{n-1} \log(uv)^{-1} \mathrm{d}u = \frac{v^{n-1}}{m^2} + \frac{v^{n-1}}{m} \log(v)^{-1}$$

$$\Rightarrow J_{m,n} = \frac{1}{m^2 n} + \frac{1}{n m^2}.$$

# Step 1: Evaluation of $I_{r,s}$

$$I_{r,s} = \sum_{k\geq 1} J_{k+r,k+s} = \sum_{n>r} J_{n,n+s-r}.$$

If $r = s$, then

$$I_{r,r} = \sum_{n>r} J_{n,n} = 2\sum_{n>r}\frac{1}{n^3} = 2\left[\zeta(3) - \sum_{n=1}^{r}\frac{1}{n^3}\right].$$

If $r \neq s$, it follows from $I_{r,s} = I_{s,r}$ that we assume WLOG $r < s$:

$$\begin{aligned} I_{r,s} &= \sum_{n>r}\frac{1}{n(n+s-r)}\left[\frac{1}{n} + \frac{1}{n+s-r}\right] \\ &= \frac{1}{s-r}\sum_{n>r}\left[\frac{1}{n} - \frac{1}{n+s-r}\right]\left[\frac{1}{n} + \frac{1}{n+s-r}\right] \end{aligned}$$

# Step 1: Evaluation of $I_{r,s}$ (continued)

$$I_{r,s} = \frac{1}{s-r}\sum_{n>r}\left[\frac{1}{n^2} - \frac{1}{(n+s-r)^2}\right] = \frac{1}{s-r}\sum_{r<n\leq s}\frac{1}{n^2}.$$

Since $(s-r)n^2 \leq s^3$, $D_s^3 I_{r,s} \in \mathbb{Z}$, where $D_n = \operatorname{lcm}(1, 2, \ldots, n)$. Conclusively, when $0 \leq r, s \leq n$, there is

$$I_{r,s} = \begin{cases} 2\left[\zeta(3) - \sum_{m=1}^{r}\frac{1}{m^3}\right] & r = s \\ K_{r,s}/D_n^3 & r \neq s \quad (K_{r,s} \in \mathbb{Z}). \end{cases}$$

# Step 1: Evaluation of $I(f)$

$$I_{r,s} = \begin{cases} 2\left[\zeta(3) - \sum_{m=1}^{r} \frac{1}{m^3}\right] & r = s \\ K_{r,s}/D_n^3 & r \neq s \quad (K_{r,s} \in \mathbb{Z}). \end{cases}$$

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ for $a_0, a_1, \ldots, a_n \in \mathbb{Z}$, then

$$\begin{aligned} I(f) &= \sum_{0\le r,s\le n} a_r a_s I_{r,s} = \sum_{r=0}^{n} a_r^2 I_{r,r} + \sum_{\substack{0\le r,s\le n \\ r\neq s}} a_r a_s I_{r,s} \\ &= 2\sum_{r=0}^{n} a_r^2 \zeta(3) - 2\sum_{r=0}^{n} a_r^2 \sum_{m=1}^{r} \frac{1}{m^3} + \sum_{\substack{0\le r,s\le n \\ r\neq s}} \frac{a_r a_s K_{r,s}}{D_n^3} \\ &= \frac{A\zeta(3) - B}{D_n^3} \quad (A \in \mathbb{Z}\setminus\{0\}, B \in \mathbb{Z}). \end{aligned}$$

# Step 3: Bounds for $I(f)$

Recall that

$$I(f) = \int_0^1 \int_0^1 \frac{\log(uv)^{-1}}{1-uv} f(u)f(v)\mathrm{d}u\mathrm{d}v,$$

so it follows from

$$\log \alpha^{-1} = \int_\alpha^1 \frac{\mathrm{d}v}{v} = \int_0^{1-\alpha} \frac{\mathrm{d}v}{1-v} = (1-\alpha)\int_0^1 \frac{\mathrm{d}z}{1-(1-\alpha)z}$$

that

$$I(f) = \int_0^1 \int_0^1 \int_0^1 \frac{f(u)f(v)}{1-(1-uv)z}\mathrm{d}u\mathrm{d}v\mathrm{d}z.$$

Let $f$ be defined as

$$f(x) = \frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n [x^n(1-x)^n].$$

# Step 3: Bounds for $I(f)$

By repeated integration by parts, there is

$$\int_0^1\int_0^1 \frac{f(u)}{1-(1-uv)z}\mathrm{d}u = \int_0^1 \frac{u^n v^n z^n (1-u)^n}{[1-(1-uv)z]^{n+1}}\mathrm{d}u. \qquad (*)$$

$$\Rightarrow I(f) = \int_0^1\int_0^1\int_0^1 \frac{u^n v^n z^n (1-u)^n f(v)}{[1-(1-uv)z]^{n+1}}\mathrm{d}u\mathrm{d}v\mathrm{d}z.$$

Let $z = (1-w)[1-(1-uv)w]^{-1}$. Then

$$I(f) = \int_0^1\int_0^1\int_0^1 (1-u)^n(1-w)^n \frac{f(v)}{1-(1-uv)w}\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

Applying $(*)$ once again, there is

# Step 3: Bounds for $I(f)$

$$I(f) = \int_0^1 \int_0^1 \int_0^1 \frac{[u(1-u)v(1-v)w(1-w)]^n}{[1-(1-uv)w]^{n+1}} \mathrm{d}u\mathrm{d}v\mathrm{d}w.$$

Since $1-(1-uv)w = (1-w) + uvw \geq 2\sqrt{1-w}\sqrt{uvw}$

$$Q = \frac{u(1-u)v(1-v)w(1-w)}{1-(1-uv)w} \leq \frac{1}{2}u^{\frac{1}{2}}(1-u)v^{\frac{1}{2}}(1-v)[w(1-w)]^{\frac{1}{2}}.$$

By differential calculus, for $0 < \alpha \leq 1$, there is

$$\sup_{u\in[0,1]} u^{\frac{1}{2}}(1-u) = \frac{2}{3\sqrt{3}}, \quad \sup_{w\in[0,1]} w(1-w) = \frac{1}{4},$$

so we have $Q \leq \frac{1}{4}\left(\frac{2}{3\sqrt{3}}\right)^2 = \frac{1}{27}$.

# Step 3: Bounds for $I(f)$

$$|I(f)| \le \left(\frac{1}{27}\right)^n \int_0^1 \int_0^1 \int_0^1 \frac{\mathrm{d}u\mathrm{d}v\mathrm{d}w}{1-(1-uv)w} = C\left(\frac{1}{27}\right)^n .$$

Let $p_1, p_2, \dots, p_{k_n}$ be all primes $\le n$. Then

$$D_n = p_1^{\lfloor \log_{p_1} n \rfloor} p_2^{\lfloor \log_{p_2} n \rfloor} \cdots p_{k_n}^{\lfloor \log_{p_{k_n}} n \rfloor} \le e^{k_n \log n}.$$

By the prime number theorem, there is

$$\lim_{n\to+\infty} \frac{k_n}{n/\log n} = 1.$$

This means for all $\varepsilon > 0$, $k_n < (1+\varepsilon)n/\log n$ for $n \ge n_0(\varepsilon)$, so $D_n < e^{(1+\varepsilon)n}$

# Finishing step

$$D_n^3|I(f)| < Ce^{3n(1+\varepsilon)}\left(\frac{1}{27}\right)^n = C\left(\frac{e^{1+\varepsilon}}{3}\right)^{3n}.$$

Choose $\varepsilon > 0$ small so that $e^{1+\varepsilon} < 3$. This ensures $D_n^3|I(f)|$ can be made arbitrarily small (e.g $< 1/k$).

If $\zeta(3) = \frac{h}{k}$, then

$$1 \leq |Ah - Bk| = kD_n^3|I(f)| < 1,$$

which is a contradiction.

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