Random matrix theory and L-functions

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The dictionary

$$\begin{tikzcd}
\boxed{\text{zeros of } \zeta(s)} \arrow[d] \arrow[r, phantom, "\Longleftrightarrow"] & \boxed{\text{eigenvalues of random matrices}} \arrow[d] \\
\boxed{\text{values of } \zeta(s)} \arrow[r, phantom, "\Longleftrightarrow"] & \boxed{\text{characteristic polynomials}}
\end{tikzcd}$$

Goal: Explain why unitary matrices predict both zero statistics and moment asymptotics for $\zeta(s)$.

Roadmap

Roadmap

1. Zeros of $\zeta(s)$ and CUE
   - Montgomery's pair correlation
   - The circular unitary ensemble
2. Values and moments of $\zeta(s)$
3. Conclusion

The local scale of zeta zeros

Assume the *Riemann Hypothesis*, so a nontrivial zero has the form

$$\rho = \frac{1}{2} + i\gamma.$$

The Riemann–von Mangoldt formula gives

$$N(T) \sim \frac{T}{2\pi} \log T.$$

Near height T, the mean gap is $\frac{2\pi}{\log T}$, so we normalize zero differences as

$$\frac{\gamma - \gamma'}{2\pi / \log T}.$$

After this normalization, do the zeros behave like independent random points?

Montgomery's pair correlation conjecture

For fixed $\alpha < \beta$, Montgomery conjectured under RH that

$$\frac{1}{N(T)}\#\left\{\gamma,\gamma' \in [0,T] : \frac{\gamma-\gamma'}{2\pi}\log T \in [\alpha,\beta]\right\}$$
$$\longrightarrow \mathbf{1}_{0\in[\alpha,\beta]} + \int_{\alpha}^{\beta}\left[1-\left(\frac{\sin \pi u}{\pi u}\right)^2\right]\mathrm{d}u.$$

Quadratic repulsion: For distinct zeros, the density is quadratic at $u = 0$.
Montgomery (1973) proved this for test functions satisfying $\operatorname{supp}\widehat{\varphi} \subset (-1,1)$.

Montgomery's investigation

In 1973, Montgomery investigated the Fourier transform of pair correlation:

$$F(\alpha, T) = \frac{1}{N(T)} \sum_{0<\gamma,\gamma' \le T} T^{i\alpha(\gamma-\gamma')} w(\gamma - \gamma').$$

Under RH, he proved $F(\alpha, T) \to \delta(\alpha) + |\alpha|$ for $|\alpha| < 1$ and conjectured

$$F(\alpha, T) \longrightarrow \delta(\alpha) + 1 - \max(1 - |\alpha|, 0).$$

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Pairing with $\widehat{\varphi}$, for φ Schwartz, gives

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{N(T)} \sum_{0<\gamma,\gamma'\le T} \varphi\left(\frac{\gamma-\gamma'}{2\pi} \log T\right) w(\gamma-\gamma') \\ &= \lim_{T\to\infty} \int_{\mathbb{R}} \widehat{\varphi}(\alpha) F(\alpha, T)\, \mathrm{d}\alpha \\ &= \varphi(0) + \int_{\mathbb{R}} \left[1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right] \varphi(u)\, \mathrm{d}u. \end{aligned}$$

Approximating $\mathbf{1}_{[\alpha,\beta]}$ by φ recovers the pair-correlation conjecture.

Fourier transform and primes

Montgomery's theorem confirms the test-function limit when $\operatorname{supp}\widehat{\varphi} \subset (-1, 1)$. Extending the support requires information about $F(\alpha, T)$ beyond $|\alpha| = 1$.

The *strong pair correlation conjecture* asserts that, for every fixed $A > 1$,

$$F(\alpha, T) \longrightarrow 1 \qquad \text{uniformly for } 1 \leq |\alpha| \leq A.$$

The Fourier transform connects the zero sum to prime sums; Parseval's identity turns this connection into the variance of primes in short intervals.

Pair correlation and twin primes

The conjectural variance of primes in short intervals is

$$J(X,H) := \int_1^X \left(\sum_{x<n\le x+H} \Lambda(n) - H \right)^2 \mathrm{d}x \sim HX \log \frac{X}{H}.$$

For fixed $A > 1$, Goldston–Montgomery (1987) showed under RH that uniform validity for $X^{\varepsilon} \le H \le X^{1-1/A-\varepsilon}$ corresponds to strong pair correlation for $1 < |\alpha| < A$, the extra range needed for $\operatorname{supp} \widehat{\varphi} \subset (-A, A)$.

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The *uniform Hardy–Littlewood prime-pair conjecture* predicts, uniformly for even $0 < h \le x^{1/2-\varepsilon}$,

$$\sum_{n\le x} \Lambda(n)\Lambda(n+h) = \mathfrak{S}(h)x + O_{\varepsilon}(x^{1/2+\varepsilon}),$$

where $\mathfrak{S}(h)$ is the singular series. The case $h = 2$ is the *twin prime conjecture*.
Inserting this into the expansion of $J(X,H)$ gives $A = 2$, hence support $(-2, 2)$.

Dyson's observation

Recall Montgomery's conjectural pair-correlation density:

$$1 - \left(\frac{\sin \pi u}{\pi u}\right)^2.$$

Dyson recognized the same density in large random unitary matrices.
Define the unitary group as

$$U(N) = \{A \in \mathbb{C}^{N\times N} : A^* A = I\}.$$

This is a compact Lie group, so it has a unique Haar probability measure known as the *circular unitary ensemble* (CUE).

The Weyl integration formula

For $A \in U(N)$, write its eigenvalues as $e^{i\theta_1}, \ldots, e^{i\theta_N}$. If h is a continuous class function on $U(N)$, then

$$\int_{U(N)} h(A) \mathrm{d}\mu_{\mathrm{CUE}}(A) = \frac{1}{(2\pi)^N N!} \int_{[0,2\pi)^N} h\left(\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_N})\right) \times \prod_{1 \le j < \ell \le N} |e^{i\theta_j} - e^{i\theta_\ell}|^2 \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_N.$$

Thus the joint eigen-angle density is

$$P_N(\theta_1, \ldots, \theta_N) = \frac{1}{(2\pi)^N N!} \prod_{1 \le j < \ell \le N} |e^{i\theta_j} - e^{i\theta_\ell}|^2.$$

The density exhibits quadratic repulsion.

The n-level correlation density

The n-level correlation measure counts ordered n-tuples of distinct eigen-angles. Its density is

$$R_{N,n}(\theta_1,\ldots,\theta_n) = \frac{N!}{(N-n)!}\int_{[0,2\pi)^{N-n}} P_N(\theta_1,\ldots,\theta_N)\mathrm{d}\theta_{n+1}\cdots\mathrm{d}\theta_N.$$

A calculation using the theory of orthogonal polynomials gives

$$R_{N,n}(\theta_1,\ldots,\theta_n) = \det\big(K_N(\theta_j,\theta_\ell)\big)_{j,\ell=1}^{n},$$

where

$$K_N(\alpha,\beta) = \frac{1}{2\pi}\frac{\sin\frac{N}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha-\beta)}.$$

The sine-kernel scaling limit

The mean eigen-angle gap is $\frac{2\pi}{N}$. At that scale,

$$\left(\frac{2\pi}{N}\right)^n R_{N,n}\left(\frac{2\pi x_1}{N},\dots,\frac{2\pi x_n}{N}\right)$$
$$\longrightarrow W_n(x_1,\dots,x_n) = \det\left(\frac{\sin \pi(x_j - x_\ell)}{\pi(x_j - x_\ell)}\right)_{j,\ell=1}^{n}.$$

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For $n = 2$,

$$W_2(x_1,x_2) = 1 - \left(\frac{\sin\pi(x_1 - x_2)}{\pi(x_1 - x_2)}\right)^2 .$$

This is exactly the density in Montgomery's conjecture. Rudnick–Sarnak (1996) verified higher correlations for restricted test functions.

What the zero statistics tell us

Local statistics of high zeros of $\zeta(s)$ are modeled by eigenvalues of large random unitary matrices.

Next: can the same matrices model the *values* of $\zeta(s)$?

Roadmap

Selberg's central limit theorem

For t chosen uniformly from $[T, 2T]$, as $T \to \infty$,

$$\frac{\log \zeta(\frac{1}{2} + it)}{\sqrt{\frac{1}{2} \log \log T}} \quad \Longrightarrow \quad X + iY,$$

where X and Y are independent standard real Gaussians.
In particular,

$$\log \left| \zeta \left(\frac{1}{2} + it \right) \right| \approx \mathcal{N} \left(0, \frac{1}{2} \log \log T \right).$$

Selberg derived the CLT from moments of $\log \zeta$, but his asymptotic covers too narrow a range of moment orders to detect large values. We therefore study

$$I_k(T) = \frac{1}{T} \int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right|^{2k} \mathrm{d}t.$$

The moment problem

The Gaussian heuristic predicts the power of the logarithm:

$$I_k(T) \asymp (\log T)^{k^2}.$$

The first two asymptotics are classical:

$$I_1(T) \sim \log T \qquad \text{(Hardy–Littlewood, 1918)}$$

$$I_2(T) \sim \frac{1}{2\pi^2}(\log T)^4. \qquad \text{(Ingham, 1926)}$$

The exponent is right, but what determines the leading constant?

Where the arithmetic factor comes from

The approximate functional equation suggests

$$\zeta\left(\frac{1}{2}+it\right)^k \approx \sum_{n\le X} \frac{\tau_k(n)}{n^{1/2+it}}.$$

Approximate orthogonality of n^{-it} then leads to

$$\sum_{n\le X} \frac{\tau_k(n)^2}{n} \sim \frac{a_k}{\Gamma(k^2+1)} (\log X)^{k^2},$$

This sum models the diagonal contribution to $I_k(T)$, with

$$a_k = \prod_p \left(1-\frac{1}{p}\right)^{k^2} \left(1+\sum_{r\ge 1} \frac{\tau_k(p^r)^2}{p^r}\right).$$

Thus we formulate the conjectural asymptotic as follows:

$$I_k(T) \sim \frac{g_k a_k}{\Gamma(k^2+1)} (\log T)^{k^2}.$$

The diagonal contribution explains a_k; random matrix theory will predict g_k.

The characteristic-polynomial model

For $g \in U(N)$, define

$$Z(g,\theta) = \det(I - ge^{-i\theta}).$$

Keating–Snaith (2000) showed that

$$\frac{\log|Z(g,\theta)|}{\sqrt{\frac{1}{2}\log N}} \implies \mathcal{N}(0,1).$$

We now match the mean gaps $\frac{2\pi}{\log T}$ for zeta zeros and $\frac{2\pi}{N}$ for eigen-angles, giving

$$N \approx \log T.$$

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With this choice, the variances also match:

$$\operatorname{Var}\log|\zeta(\tfrac{1}{2}+it)| \sim \frac{1}{2}\log\log T, \qquad \operatorname{Var}\log|Z| \sim \frac{1}{2}\log N.$$

Finite-height evidence

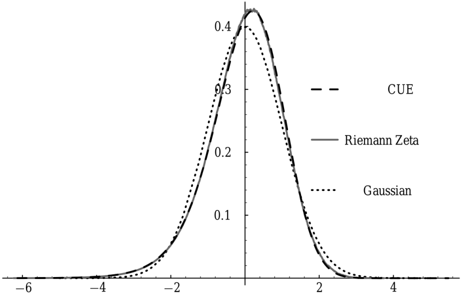

Figure: $N = 42$ CUE and ζ near $T = 1.52 \times 10^{19}$ (Keating–Snaith, 2000).

At finite height, CUE fits better than Selberg's Gaussian limit.

The random matrix moment formula

Weyl's integration formula, followed by Selberg's integral, gives

$$\begin{aligned} J_k(N) &= \int_{U(N)} |\det(I-g)|^{2k} \mathrm{d}g \\ &= \frac{1}{(2\pi)^N N!} \int_{[0,2\pi)^N} \prod_{j=1}^{N} |1-e^{i\theta_j}|^{2k} \prod_{1\le j<\ell\le N} |e^{i\theta_j}-e^{i\theta_\ell}|^2 \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_N \\ &= \prod_{r=1}^{N} \frac{\Gamma(r)\Gamma(2k+r)}{\Gamma(k+r)^2}. \end{aligned}$$

Using the Barnes G-function,

$$J_k(N) \sim \frac{G(1+k)^2}{G(1+2k)} N^{k^2} = \Gamma(1+k^2) \frac{G(1+k)^2}{G(1+2k)} \cdot \frac{N^{k^2}}{\Gamma(1+k^2)}$$

The Keating–Snaith conjecture

Keating–Snaith (2000) predict

$$g_k = \Gamma(k^2+1)\frac{G(1+k)^2}{G(1+2k)} \qquad \left(= (k^2)!\prod_{j=0}^{k-1}\frac{j!}{(j+k)!} \quad \text{if } k \in \mathbb{N}\right).$$

Combining the matrix factor with the arithmetic factor gives

$$I_k(T) \sim \frac{g_k a_k}{\Gamma(k^2+1)}(\log T)^{k^2}.$$

k	g_k	status
1	1	Hardy–Littlewood (1918)
2	2	Ingham (1926)
$k > 2$	explicit Barnes-G value	conjectural asymptotic

The hybrid Euler–Hadamard model

The vanilla matrix model does not explain a_k. Gonek–Hughes–Keating (2007) split zeta into two pieces:

$$\zeta(s) \approx \underbrace{\prod_{p \le X} (1 - p^{-s})^{-1}}_{P_X(s):\ \text{nearby primes}} \times \underbrace{\prod_{\substack{\rho = \frac{1}{2} + i\gamma \\ |t-\gamma| \le 1/\log X}} \left[(s-\rho) e^{\gamma} \log X \right]}_{Z_X(s):\ \text{nearby zeros}} .$$

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The *splitting conjecture* asserts, roughly,

$$\text{moment}(\zeta) \sim \text{moment}(P_X) \times \text{moment}(Z_X).$$

P_X explains a_k. Z_X accounts for g_k.

Lower bounds for zeta moments

The conjectural order of magnitude requires

$$I_k(T) \geq c_k (\log T)^{k^2}.$$

Work	Range of k
Ramachandra (1978, 1980)	positive half-integers
Heath-Brown (1981)	positive rational numbers
Radziwiłł–Soundararajan (2013)	real $k \geq 1$ with $c_k = e^{-30k^4}$
Heap–Soundararajan (2020)	real $0 < k \leq 1$

Together these results prove the conjectural lower-bound order for every real $k > 0$.

Upper bounds for zeta moments

The conjectural upper bound is

$$I_k(T) \ll_k (\log T)^{k^2}.$$

- Soundararajan (2009) proved the exponent $k^2 + \varepsilon$ under RH.
- Harper (2013) removed ε under RH.
- Heap–Radziwiłł–Soundararajan (2019) proved the sharp bound unconditionally for $0 < k \leq 2$.

Therefore the sharp upper bound is unconditional for $0 < k \leq 2$ and follows from RH for $k \geq 2$.

Full asymptotic formulas remain known only for $k = 1, 2$.

Roadmap

Conclusion

We just developed a reusable recipe for general L-functions:

1. Normalize zeros by mean spacing and match the correlation statistics with random matrix models.
2. Compare L-values with random characteristic polynomials.
3. Evaluate the Haar integral to obtain the universal factor and log exponent.
4. Restore the arithmetic Euler product coming from the primes.

$$\boxed{\text{RMT coefficient} \times \text{arithmetic Euler product} \times \text{power of } \log T}$$

Moments of $L(1/2, \chi_d)$

Let

$$\mathcal{D}(X) = \{d : d \text{ a fundamental discriminant, } |d| \leq X\}.$$

Then the recipe gives the following Keating–Snaith conjecture:

$$\frac{1}{|\mathcal{D}(X)|} \sum_{d \in \mathcal{D}(X)} L\left(\frac{1}{2}, \chi_d\right)^k \sim 2^{k^2/2} \frac{G(1+k)\sqrt{\Gamma(1+k)}}{\sqrt{G(1+2k)\Gamma(1+2k)}}$$

$$\times \prod_p \left\{ \frac{\left(1 - \frac{1}{p}\right)^{k(k+1)/2}}{p+1} \left(1 + \frac{p}{2}\left[\left(1 + \frac{1}{\sqrt{p}}\right)^{-k} + \left(1 - \frac{1}{\sqrt{p}}\right)^{-k}\right]\right) \right\}$$

$$\times (\log \sqrt{X})^{k(k+1)/2}$$

for fixed $k \geq 0$.

If time permits: families of L-functions

From unitary statistics for $\zeta(s)$ to Katz–Sarnak symmetry types

One L-function versus a family

For an individual primitive $L(s,\pi)$:

$$\text{high zeros} \quad \longleftrightarrow \quad U(N) \text{ eigenvalues.}$$

For a family $\mathcal{F}$, study low-lying zeros

$$\frac{1}{2} + i\gamma_f, \qquad \frac{\gamma_f \log C_f}{2\pi} = O(1),$$

near the central point.
Katz–Sarnak predict that the family has a symmetry type

$$U(N), \quad USp(2N), \quad SO(2N), \quad SO(2N+1), \quad \text{or} \quad O(N).$$

Low-lying zeros are sensitive to self-duality and to the sign of the functional equation—information invisible in the universal high-zero limit.

The one-level density test

For a Schwartz function φ, define

$$D(f;\varphi) = \sum_{\gamma_f} \varphi\left(\frac{\gamma_f \log C_f}{2\pi}\right).$$

Average over the family:

$$E(\mathcal{F}(Q);\varphi) = \frac{1}{|\mathcal{F}(Q)|} \sum_{f\in\mathcal{F}(Q)} D(f;\varphi).$$

The symmetry type G is identified by the limit

$$E(\mathcal{F}(Q);\varphi) \longrightarrow \int_{\mathbb{R}} \varphi(x) D_G(x)\mathrm{d}x.$$

On the matrix side, D_G is the microscopic one-level density of eigenvalues near 1.

The classical one-level densities

$$
\begin{aligned}
D_U(x) &= 1, \\
D_{USp}(x) &= 1 - \frac{\sin 2\pi x}{2\pi x}, \\
D_O(x) &= 1 + \frac{1}{2}\delta(x), \\
D_{SO(\text{even})}(x) &= 1 + \frac{\sin 2\pi x}{2\pi x}, \\
D_{SO(\text{odd})}(x) &= \delta(x) + 1 - \frac{\sin 2\pi x}{2\pi x}.
\end{aligned}
$$

Symmetry	Behavior at the central point
Unitary	no special bias
Symplectic	repulsion from zero
Even orthogonal	attraction toward zero
Odd orthogonal	forced central zero

Examples of symmetry types

Family	Symmetry type
primitive characters χ mod p, $p \to \infty$	Unitary
quadratic characters χ_d, $\lvert d\rvert \le X$, d fundamental	Symplectic
weight-k Hecke eigenforms on $SL_2(\mathbb{Z})$	Orthogonal
$k \equiv 0 \pmod 4$	$SO(2N)$
$k \equiv 2 \pmod 4$	$SO(2N+1)$

The orthogonal distinction records the sign $(-1)^{k/2}$ of the functional equation.
Katz–Sarnak (1999) proved this philosophy broadly over function fields; over number fields, results such as Iwaniec–Luo–Sarnak (2000) require support restrictions.

Symmetry predicts moment growth

For a self-dual family with non-negative central values, model

$$\frac{1}{|\mathcal{F}(Q)|}\sum_{f\in\mathcal{F}(Q)} L\left(\frac{1}{2},f\right)^k$$

by

$$\int_{G(N)} \det(I-g)^k \mathrm{d}g.$$

As in the hybrid Euler–Hadamard model, we introduce an Euler-product factor to restore the arithmetic. The complete conjecture is

$$\text{RMT coefficient} \times \text{arithmetic Euler product} \times \text{power of } \log Q.$$

Example: quadratic Dirichlet L-functions

Let

$$\mathcal{D}(X) = \{d : d \text{ a fundamental discriminant, } |d| \le X\}.$$

The family $\{L(s, \chi_d)\}_{d \in \mathcal{D}(X)}$ is symplectic, hence

$$L\left(\frac{1}{2}, \chi_d\right) \quad \longleftrightarrow \quad \det(I - g), \qquad g \in USp(2N), \quad N \approx \frac{1}{2} \log X.$$

Both models predict

$$\log L\left(\frac{1}{2}, \chi_d\right) \approx \mathcal{N}\left(\frac{1}{2} \log\log X,\ \log\log X\right).$$

Thus the exponent of log here is

$$\frac{k(k+1)}{2} \qquad \text{instead of the unitary exponent} \qquad k^2.$$

Moments of $L(1/2, \chi_d)$

For this family, the recipe gives the following Keating–Snaith conjecture:

$$\frac{1}{|\mathcal{D}(X)|} \sum_{d \in \mathcal{D}(X)} L\left(\frac{1}{2}, \chi_d\right)^k \sim 2^{k^2/2} \frac{G(1+k)\sqrt{\Gamma(1+k)}}{\sqrt{G(1+2k)\Gamma(1+2k)}} \times \prod_p \left\{ \frac{\left(1 - \frac{1}{p}\right)^{k(k+1)/2}}{p+1} \left(1 + \frac{p}{2}\left[\left(1 + \frac{1}{\sqrt{p}}\right)^{-k} + \left(1 - \frac{1}{\sqrt{p}}\right)^{-k}\right]\right) \right\} \times (\log \sqrt{X})^{k(k+1)/2}$$

for fixed $k \geq 0$.