Random matrix theory and L-functions

Travor Liu

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Abstract

In 1973, Montgomery conjectured that under the Riemann Hypothesis, the number of zeros $\frac{1}{2} + i\gamma$, $\frac{1}{2} + i\gamma'$ of $\zeta(s)$ satisfying

$$\frac{\gamma - \gamma'}{2\pi/\log T} \in [\alpha, \beta], \qquad \gamma, \gamma' \in [0, T]$$

as $T \to +\infty$ is

$$\sim \frac{T}{2\pi} \log T \left\{ \int_\alpha^\beta \left[1 - \left(\frac{\sin \pi u}{\pi u} \right)^2 \right] \mathrm{d}u + \mathbf{1}_{0 \in [\alpha, \beta]} \right\}.$$

Dyson pointed out that this is related to the eigenvalue distribution of random unitary matrices. Not only does this connection allow us to better understand the zeros of $\zeta(s)$ but also the distribution of its values on $\mathrm{Re}(s) = \frac{1}{2}$.

This article surveys this connection and explains how Keating–Snaith (2000) [19] used it to formulate a conjectured asymptotic for the moments

$$I_k(T) = \frac{1}{T} \int_0^T \left| \zeta\left(\frac{1}{2} + it \right) \right|^{2k} \mathrm{d}t.$$

We also describe extensions of these ideas to families of L-functions and their symmetry types.

Contents

1 Montgomery's Conjecture

Let $N(T)$ be the number of nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \le T$. It is known that

$$N(T) \sim \frac{T}{2\pi} \log T,$$

which indicates that the mean gap between the ordinates γ of zeros with $\gamma \in [0,T]$ is $\sim \frac{2\pi}{\log T}$. Thus, to understand the correlation between two zeros, the natural object to study is the normalized difference $\frac{\gamma-\gamma'}{2\pi}\log T$.

In 1973, Montgomery [21] investigated the Fourier transform of pair correlation:

$$F(\alpha,T) = \frac{1}{N(T)} \sum_{0<\gamma,\gamma'\le T} T^{i\alpha(\gamma-\gamma')} w(\gamma-\gamma'),$$

where w is a suitable smooth weight with $w(0) = 1$. Assuming RH, he computed that $F(\alpha,T) \to \delta(\alpha) + |\alpha|$ for $\alpha \in (-1,1)$ in the distributional sense. For $|\alpha| \ge 1$, Montgomery conjectured that the limit is 1. In other words, the full conjecture is

$$\lim_{T\to\infty} F(\alpha,T) = \delta(\alpha) + 1 - \max(1-|\alpha|,0).$$

Pairing the conjectured distributional limit with $\widehat{\varphi}$ for φ Schwartz,

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{N(T)} \sum_{0<\gamma,\gamma'\le T} \varphi\left(\frac{\gamma-\gamma'}{2\pi/\log T}\right) w(\gamma-\gamma') &= \lim_{T\to\infty} \int_{\mathbb{R}} \widehat{\varphi}(\alpha) F(\alpha,T)\mathrm{d}\alpha \\ &= \varphi(0) + \int_{\mathbb{R}} \left[1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right] \varphi(u)\mathrm{d}u. \end{aligned} \tag{1.1}$$

Letting φ approximate $\mathbf{1}_{[\alpha,\beta]}$ recovers the pair correlation conjecture.[1]

$$\lim_{T\to\infty} \frac{\#\left\{\gamma,\gamma' \in [0,T] : \frac{\gamma-\gamma'}{2\pi}\log T \in [\alpha,\beta]\right\}}{N(T)} = \mathbf{1}_{0\in[\alpha,\beta]} + \int_\alpha^\beta \left[1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right] \mathrm{d}u.$$

In the language of test functions, Montgomery's work is essentially confirming the limit (1.1) when $\operatorname{supp}\widehat{\varphi} \subset (-1,1)$. Using the explicit formula and Parseval's formula, Goldston–Montgomery (1987) [5] showed that the pair correlation of zeros is closely related to correlations of primes in short intervals. For example, a sufficiently strong form of the twin prime conjecture extends (1.1) to test functions φ with $\operatorname{supp}\widehat{\varphi} \subset (-2,2)$.

Dyson noted that the function

$$1 - \left(\frac{\sin \pi u}{\pi u}\right)^2$$

in the integral agrees with the 2-level correlation density of the eigen-angles of large random unitary matrices. We now survey what this means.

2 Circular Unitary Ensemble

For $A \in \mathbb{C}^{N\times N}$, let A^* denote its conjugate transpose. Regard $\mathbb{C}^{N\times N}$ as a real inner-product space with the Hilbert–Schmidt inner product

$$\langle X,Y\rangle_{\mathrm{HS}} = \operatorname{Re}\operatorname{Tr}(X^*Y), \qquad \|X\|_{\mathrm{HS}}^2 = \operatorname{Tr}(X^*X).$$

Define the unitary group as

$$U(N) = \{A \in \mathbb{C}^{N\times N} : A^*A = I\} \subset GL_N(\mathbb{C}).$$

The map $A \mapsto A^*A$ is continuous, so $U(N)$ is closed in $GL_N(\mathbb{C})$. Since $U(N)$ is also a subgroup, the closed subgroup theorem shows that it is a Lie group. Moreover,

$$U(N) \subset \{A \in \mathbb{C}^{N\times N} : \|A\|_{\mathrm{HS}} = \sqrt{N}\},$$

because $\|A\|_{\mathrm{HS}}^2 = \operatorname{Tr}(A^*A) = N$ for $A \in U(N)$. Thus, it follows from Heine–Borel that $U(N)$ is also compact. Consequently, it has a unique invariant probability measure known as the *circular unitary ensemble* (CUE) measure μ_{CUE}.

[1] On a fixed normalized interval, $\gamma - \gamma' = O(1/\log T)$, so $w(\gamma-\gamma') = 1 + o(1)$ uniformly; hence the weight may be omitted.

2.1 Distribution of Eigenvalues

By the spectral theorem, each $A \in U(N)$ is conjugate in $U(N)$ to some

$$E = \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_N}),$$

where $\theta_j \in [0, 2\pi)$ are called the eigen-angles of A. A continuous *class function* h on $U(N)$ satisfies $h(RAR^*) = h(A)$ and therefore depends only on the unordered eigenvalues of A. Conversely, every continuous symmetric function of the eigenvalues defines a class function. Thus, the joint eigenvalue distribution is determined by the integrals

$$\int_{U(N)} h(A) \mathrm{d}\mu_{\mathrm{CUE}}(A)$$

for all continuous class functions h. We derive a formula for these integrals by expressing a volume form in eigenvalue–eigenvector coordinates; the joint density will then be the symmetric density that makes the formula hold for every h.

For a tangent vector $\mathrm{d}A$ at A, differentiating $A^*A = I$ gives

$$(\mathrm{d}A)^*A + A^*\mathrm{d}A = 0.$$

Thus, the matrix-valued 1-form

$$\Xi = A^*\mathrm{d}A$$

is skew-Hermitian. The map $\mathrm{d}A \mapsto A^*\mathrm{d}A$ is a linear isomorphism from the tangent space at A to the real vector space of skew-Hermitian matrices. Therefore, the real 1-forms $-i\Xi_{jj}$ and, for $j < \ell$, $\operatorname{Re}\Xi_{j\ell}$ and $\operatorname{Im}\Xi_{j\ell}$ form a basis of the cotangent space. Their wedge product

$$\omega_{U(N)} = \bigwedge_{j=1}^{N}(-i\Xi_{jj}) \wedge \bigwedge_{1\le j<\ell\le N}\left(\operatorname{Re}\Xi_{j\ell} \wedge \operatorname{Im}\Xi_{j\ell}\right) \tag{2.1}$$

is consequently a nowhere-vanishing differential form of degree N^2.

This form gives the CUE volume. Indeed, for fixed $P, Q \in U(N)$, the change $A \mapsto PAQ$ sends Ξ to

$$(PAQ)^*\mathrm{d}(PAQ) = Q^*\Xi Q.$$

The linear map $X \mapsto Q^*XQ$ preserves the inner product $\operatorname{Re}\operatorname{Tr}(X^*Y)$ on skew-Hermitian matrices, so its real Jacobian determinant has absolute value 1. Hence $A \mapsto PAQ$ preserves the absolute volume determined by (2.1). After normalization, this volume therefore gives the CUE probability measure.

Suppose now that the eigenvalues of A are distinct. Taking normalized eigenvectors of A as the columns of $R \in U(N)$ gives the unitary diagonalization

$$A = RER^*.$$

Since R is unitary, the same calculation used for $\Xi = A^*\mathrm{d}A$ shows that $\Omega = R^*\mathrm{d}R$ is skew-Hermitian.

Differentiating $A = RER^*$ and using $\mathrm{d}R^*R = -R^*\mathrm{d}R = -\Omega$ gives

$$R^*(\mathrm{d}A)R = \mathrm{d}E + \Omega E - E\Omega.$$

Consequently,

$$R^*\Xi R = E^{-1}\mathrm{d}E + E^{-1}\Omega E - \Omega.$$

Replacing Ξ by $R^*\Xi R$ in (2.1) changes the form at most by a sign, by the same determinant calculation used above. The entries of $R^*\Xi R$ are

$$(R^*\Xi R)_{jj} = i\,\mathrm{d}\theta_j,$$
$$(R^*\Xi R)_{j\ell} = \left(e^{i(\theta_\ell-\theta_j)} - 1\right)\Omega_{j\ell}, \qquad j \neq \ell.$$

If a complex 1-form β is multiplied by a complex-valued smooth function λ, then

$$\operatorname{Re}(\lambda\beta) \wedge \operatorname{Im}(\lambda\beta) = |\lambda|^2 \operatorname{Re}\beta \wedge \operatorname{Im}\beta.$$

For the scalar appearing above,

$$\left|e^{i(\theta_\ell-\theta_j)}-1\right|^2=|e^{i\theta_\ell}-e^{i\theta_j}|^2.$$

The diagonalization is unchanged if each column of R is multiplied by an independent complex number of absolute value 1. We therefore regard two such matrices R as the same eigenvector choice. The space of eigenvector choices has dimension $N^2-N=N(N-1)$, and the form

$$\omega_R=\bigwedge_{1\le j<\ell\le N}\left(\operatorname{Re}\Omega_{j\ell}\wedge\operatorname{Im}\Omega_{j\ell}\right).$$

is a nowhere-vanishing form of top degree on this space. It is unchanged by the choices of column phases and depends only on the eigenvectors. Taking the wedge product of all the diagonal and off-diagonal entries gives, up to a sign independent of the eigen-angles,

$$\omega_{U(N)}=\prod_{1\le j<\ell\le N}|e^{i\theta_j}-e^{i\theta_\ell}|^2\mathrm{d}\theta_1\wedge\cdots\wedge\mathrm{d}\theta_N\wedge\omega_R. \tag{2.2}$$

Let ν be the volume measure determined by $\omega_{U(N)}$, and let V_{vec} denote the total absolute volume of the eigenvector choices determined by ω_R. For a class function h, we have $h(RER^*)=h(E)$. Moreover, permuting the eigenvalues and the corresponding eigenvectors leaves A unchanged, so labeled eigen-angles count each matrix with distinct eigenvalues $N!$ times. Therefore, integrating (2.2) gives

$$\int_{U(N)}h(A)\mathrm{d}\nu(A)=\frac{V_{\mathrm{vec}}}{N!}\int_{[0,2\pi)^N}h(E)\prod_{1\le j<\ell\le N}|e^{i\theta_j}-e^{i\theta_\ell}|^2\mathrm{d}\theta_1\cdots\mathrm{d}\theta_N.$$

Taking $h=1$ gives $\nu(U(N))=\frac{V_{\mathrm{vec}}}{N!}C_N$, where C_N is the integral on the right without $h(E)$. Thus, after normalizing ν to obtain μ_{CUE},

$$\int_{U(N)}h(A)\mathrm{d}\mu_{\mathrm{CUE}}(A)=\frac{1}{C_N}\int_{[0,2\pi)^N}h(E)\prod_{1\le j<\ell\le N}|e^{i\theta_j}-e^{i\theta_\ell}|^2\mathrm{d}\theta_1\cdots\mathrm{d}\theta_N.$$

It remains to determine C_N. Vandermonde's determinant formula gives

$$\prod_{1\le j<\ell\le N}(e^{i\theta_\ell}-e^{i\theta_j})=\det(e^{i(j-1)\theta_\ell})_{j,\ell}.$$

Expanding this determinant and its complex conjugate, then using

$$\int_0^{2\pi}e^{im\theta}\mathrm{d}\theta=2\pi\mathbf{1}_{m=0},$$

we obtain

$$\begin{aligned}C_N&=\int_{[0,2\pi]^N}\prod_{1\le j<\ell\le N}|e^{i\theta_j}-e^{i\theta_\ell}|^2\mathrm{d}\theta_1\cdots\mathrm{d}\theta_N\\&=(2\pi)^N\sum_{\sigma,\tau\in S_N}\operatorname{sgn}(\sigma\tau)\prod_{j=1}^N\mathbf{1}_{\sigma(j)=\tau(j)}=(2\pi)^NN!.\end{aligned}$$

Substituting this value of C_N into the class-function identity above gives the *Weyl integration formula* for $U(N)$:

$$\int_{U(N)}h(A)\mathrm{d}\mu_{\mathrm{CUE}}(A)=\frac{1}{(2\pi)^NN!}\int_{[0,2\pi)^N}h(E)\prod_{1\le j<\ell\le N}|e^{i\theta_j}-e^{i\theta_\ell}|^2\mathrm{d}\theta_1\cdots\mathrm{d}\theta_N. \tag{2.3}$$

Matrices with repeated eigenvalues form a set of volume zero. Since $h(E)$ may be any continuous symmetric function of the eigen-angles, the identity (2.3) shows that their joint density under CUE is [20]

$$P_N(\theta_1,\theta_2,\ldots,\theta_N)=\frac{1}{(2\pi)^NN!}\prod_{1\le j<\ell\le N}|e^{i\theta_j}-e^{i\theta_\ell}|^2 \tag{2.4}$$

with respect to $\mathrm{d}\theta_1\mathrm{d}\theta_2\cdots\mathrm{d}\theta_N$.

2.2 Correlation of Eigenvalues

By the *n-level correlation measure*, we mean the counting measure of ordered tuples $(\theta_{j_1}, \theta_{j_2}, \ldots, \theta_{j_n})$ with $1 \leq j_1, j_2, \ldots, j_n \leq N$ distinct. Its total mass is $\frac{N!}{(N-n)!}$, and its density is

$$R_{N,n}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{N!}{(N-n)!} \int_{[0,2\pi]^{N-n}} P_N(\theta_1, \ldots, \theta_N) \mathrm{d}\theta_{n+1} \cdots \mathrm{d}\theta_N. \tag{2.5}$$

Using Vandermonde's determinant formula and the Leibniz expansion, we see that for complex numbers $z_1, \ldots, z_N$,

$$\prod_{1 \leq j < \ell \leq N} |z_j - z_\ell|^2 = \sum_{\sigma, \tau \in S_N} \operatorname{sgn}(\sigma\tau) \prod_{j=1}^{N} z_j^{\sigma(j)-1} \bar{z}_j^{\tau(j)-1},$$

so (2.5) becomes

$$R_{N,n}(\theta_1, \ldots, \theta_n) = \frac{1}{(2\pi)^n (N-n)!} \sum_{\substack{\sigma, \tau \in S_N \\ \sigma(k) = \tau(k)\ \forall k > n}} \operatorname{sgn}(\sigma\tau) \prod_{j=1}^{n} e^{i\theta_j[\sigma(j) - \tau(j)]}.$$

Because $\sigma(k) = \tau(k)$ for $k > n$, $\tau(1), \tau(2), \ldots, \tau(n)$ is a reordering of $\sigma(1), \sigma(2), \ldots, \sigma(n)$. Equivalently, both are reorderings of $k_1 + 1, k_2 + 1, \ldots, k_n + 1$ for some $0 \leq k_1 < k_2 < \cdots < k_n < N$. This allows us to rewrite $R_{N,n}$ as

$$\frac{1}{(2\pi)^n} \sum_{0 \leq k_1 < k_2 < \cdots < k_n < N} \sum_{g, h \in S_n} \operatorname{sgn}(gh) \prod_{j=1}^{n} e^{i\theta_j (k_{g(j)} - k_{h(j)})} = \frac{1}{(2\pi)^n} \sum_{0 \leq k_1 < k_2 < \cdots < k_n < N} \left| \det(e^{i\theta_j k_\ell})_{j,\ell} \right|^2.$$

Lemma 2.1 (Gram's identity)**.** *Let $v_1, v_2, \ldots, v_n \in \mathbb{C}^N$ with $n \leq N$, and write $v_j = (v_j(1), v_j(2), \ldots, v_j(N))$. With*

$$\langle v_j, v_\ell \rangle = \sum_{k=1}^{N} v_j(k) \overline{v_\ell(k)},$$

we have

$$\det(\langle v_j, v_\ell \rangle)_{1 \leq j, \ell \leq n} = \sum_{1 \leq k_1 < k_2 < \cdots < k_n \leq N} \left| \det(v_j(k_\ell))_{1 \leq j, \ell \leq n} \right|^2.$$

Apply Lemma 2.1 with

$$v_j = \frac{1}{\sqrt{2\pi}} \left(1, e^{i\theta_j}, e^{2i\theta_j}, \ldots, e^{(N-1)i\theta_j} \right).$$

The sum of squared minors above is then $\det(\langle v_j, v_\ell \rangle)_{j,\ell}$. By the geometric sum formula,

$$\begin{aligned}
\langle v_j, v_\ell \rangle &= \frac{1}{2\pi} \sum_{k=0}^{N-1} e^{ik(\theta_j - \theta_\ell)} \\
&= \frac{1}{2\pi} e^{\frac{i(N-1)(\theta_j - \theta_\ell)}{2}} \frac{\sin \frac{N}{2}(\theta_j - \theta_\ell)}{\sin \frac{1}{2}(\theta_j - \theta_\ell)}.
\end{aligned}$$

Define

$$K_N(\alpha, \beta) = \frac{1}{2\pi} \frac{\sin \frac{N}{2}(\alpha - \beta)}{\sin \frac{1}{2}(\alpha - \beta)},$$

with $K_N(\alpha, \alpha) = \frac{N}{2\pi}$ by continuity. Set

$$\mathcal{G} = (\langle v_j, v_\ell \rangle)_{j,\ell}, \qquad \mathcal{K} = (K_N(\theta_j, \theta_\ell))_{j,\ell}, \qquad D = \operatorname{diag}\left(e^{\frac{i(N-1)\theta_1}{2}}, \ldots, e^{\frac{i(N-1)\theta_n}{2}} \right).$$

Then $\mathcal{G} = D\mathcal{K}D^*$, and hence

$$\begin{aligned}
\det \mathcal{G} &= \det D \det \mathcal{K} \det D^* \\
&= |\det D|^2 \det \mathcal{K} = \det \mathcal{K},
\end{aligned}$$

since D is unitary. Therefore,

$$R_{N,n}(\theta_1, \ldots, \theta_n) = \det(K_N(\theta_j, \theta_\ell))_{j,\ell}.$$

The mean gap between N angles is $\frac{2\pi}{N}$. Thus, the normalized n-level correlation density is

$$\left(\frac{2\pi}{N}\right)^n R_{N,n}\left(\frac{2\pi}{N}x_1, \ldots, \frac{2\pi}{N}x_n\right) = \det\left(\frac{1}{N}\frac{\sin \pi(x_j - x_\ell)}{\sin \frac{\pi}{N}(x_j - x_\ell)}\right)_{1 \le j,\ell \le n}.$$

Taking $N \to +\infty$, we obtain the *scaling limit* of the CUE n-level correlations:

$$W_n(x_1, \ldots, x_n) = \det\left(\frac{\sin \pi(x_j - x_\ell)}{\pi(x_j - x_\ell)}\right)_{1 \le j,\ell \le n}. \tag{2.6}$$

In particular, when $n = 2$,

$$W_2(x_1, x_2) = 1 - \left(\frac{\sin \pi(x_1 - x_2)}{\pi(x_1 - x_2)}\right)^2$$

appears in Montgomery's conjecture. The formula (2.6) allows us to generalize Montgomery's conjecture to correlations of $n \ge 3$ zeros. Rudnick–Sarnak [25] verified (2.6) on zeros of $\zeta(s)$ on a special class of test functions under RH.

3 Values of $\zeta(s)$ on $\mathrm{Re}(s) = \frac{1}{2}$

For a random variable X, its distribution can be studied by looking at the moments $m_k = \mathbb{E}[X^k]$. Sometimes, moments uniquely determine a distribution; for example, Gaussian moments determine a Gaussian. Upper bounds for moments are related to large deviations. For these reasons, moment calculations are very common in analytic number theory.

3.1 Selberg's Central Limit Theorem

In the 1940s, Selberg computed the moments of $\mathrm{Re} \log \zeta(\frac{1}{2} + it)$ and $\mathrm{Im} \log \zeta(\frac{1}{2} + it)$ [27, 28, 31], effectively showing that after division by $\sqrt{\frac{1}{2} \log \log T}$, they jointly tend to independent standard real Gaussians. That is, for every box $B \subset \mathbb{C}$,

$$\frac{1}{T} \operatorname{meas}\left\{t \in [T, 2T] : \frac{\log \zeta(\frac{1}{2} + it)}{\sqrt{\frac{1}{2} \log \log T}} \in B\right\} \longrightarrow \frac{1}{2\pi} \int_{x+iy \in B} e^{-\frac{1}{2}(x^2+y^2)} \mathrm{d}x\mathrm{d}y.$$

Selberg's theorem describes the bulk distribution of $\log |\zeta(\frac{1}{2} + it)|$, but it does not control the rare large values that can dominate its moments. To investigate these tails, we study

$$I_k(T) = \frac{1}{T} \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} \mathrm{d}t. \tag{3.1}$$

3.2 Moments of $|\zeta(\frac{1}{2} + it)|$

If X is a Gaussian variable with mean μ and variance σ^2, then

$$\mathbb{E}[e^{tX}] = e^{t\mu + t^2\sigma^2/2}.$$

Plugging $\mu = 0$, $\sigma^2 = \frac{1}{2} \log \log T$, and $t = 2k$ suggests that

$$I_k(T) \sim (\log T)^{k^2}. \tag{3.2}$$

Hardy–Littlewood (1918) [7] proved $I_1(T) \sim \log T$. Ingham (1926) [14] proved

$$I_2(T) \sim \frac{1}{2\pi^2} (\log T)^4.$$

While the magnitude agrees, the leading coefficients are inconsistent at $k = 2$, so we should weaken (3.2) to

$$I_k(T) \sim C_k (\log T)^{k^2}$$

for some $C_k > 0$.

When $k \geq 1$ is an integer, the approximate functional equations suggest $\zeta(\frac{1}{2} + it)^k$ can be estimated by

$$\sum_{n \leq X} \frac{\tau_k(n)}{n^{1/2+it}},$$

and the approximate orthogonality of $\{n^{-it}\}_{t\in[0,T]}$ suggests that the main term of $I_k(T)$ should be related to

$$\sum_{n \leq X} \frac{\tau_k(n)^2}{n} \sim \frac{a_k}{\Gamma(k^2+1)} (\log X)^{k^2},$$

with

$$a_k = \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \left(1 + \sum_{r=1}^{\infty} \frac{\tau_k(p^r)^2}{p^r}\right).$$

Thus, we should write

$$C_k = \frac{g_k a_k}{\Gamma(k^2+1)}.$$

In this language, Hardy–Littlewood's result says $g_1 = 1$, and Ingham's says $g_2 = 2$.

By relating $\zeta(\frac{1}{2} + it)$ and the characteristic polynomials of random unitary matrices, Keating–Snaith (2000) [19] were able to formulate conjectures for g_k for all $k > 0$.

3.3 Characteristic Polynomials in CUE

The normalized characteristic polynomial for $g \in U(N)$ is

$$Z(g, \theta) = \det(I - g e^{-i\theta}).$$

By the method of Fourier transform, Keating–Snaith [19] showed that $\operatorname{Re}\log Z(g,\theta) = \log|Z(g,\theta)|$, after division by $\sqrt{\frac{1}{2}\log N}$, tends to a standard real Gaussian. Recall that the mean gap for zeros of $\zeta(s)$ is $\frac{2\pi}{\log T}$, while that for eigen-angles is $\frac{2\pi}{N}$. Taking $N \approx \log T$ suggests a limiting distribution matching Selberg's CLT for $\log|\zeta(\frac{1}{2}+it)| = \operatorname{Re}\log\zeta(\frac{1}{2}+it)$. Thus, we should believe that the distribution of $\{\log|\zeta(\frac{1}{2}+it)|\}_{t\in[0,T]}$ is similar to that of $\{\operatorname{Re}\log Z(g,\theta)\}_{g\in U(\log T)}$. Numerically, the distribution of $\log|\zeta(\frac{1}{2}+it)|$ is much closer to that of $\operatorname{Re}\log Z(g,\theta)$ than to that of a Gaussian when T is small, as shown in Figure 1.

Therefore, we should gain insights by computing

$$J_k(N) = \int_{g\in U(N)} \left(\frac{1}{2\pi}\int_0^{2\pi} |Z(g,\theta)|^{2k} \mathrm{d}\theta\right) \mathrm{d}\mu_{\mathrm{CUE}}(g).$$

By invariance of CUE under $g \mapsto g e^{-i\theta}$, and since $g \mapsto |\det(I-g)|^{2k}$ is a class function, (2.3) gives

$$\begin{aligned} J_k(N) &= \int_{g\in U(N)} |\det(I-g)|^{2k} \mathrm{d}\mu_{\mathrm{CUE}}(g) \\ &= \frac{1}{(2\pi)^N N!} \int_{[0,2\pi]^N} \prod_{j=1}^{N} |1 - e^{i\theta_j}|^{2k} \prod_{1\leq j<\ell\leq N} |e^{i\theta_j} - e^{i\theta_\ell}|^2 \mathrm{d}\theta_1 \mathrm{d}\theta_2 \cdots \mathrm{d}\theta_N. \end{aligned}$$

To continue from here, we need a unit-circle analogue of Selberg's integral [26].

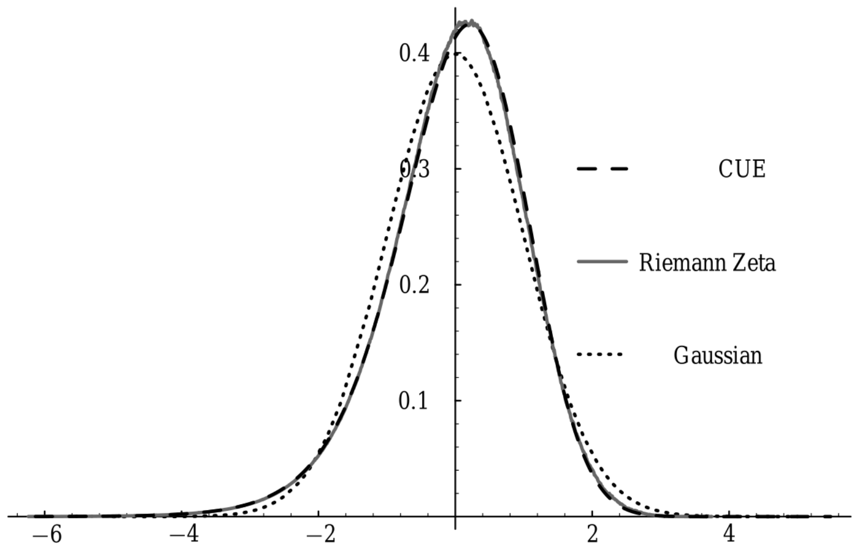

Figure 1: Comparison, after scaling to unit variance, of the value distributions of $\operatorname{Re}\log Z$ for CUE with $N = 42$, $\operatorname{Re}\log \zeta(\frac{1}{2} + it)$ near the 10^{20}th zero, and a standard Gaussian. Reproduced from Keating–Snaith [19, Fig. 1].

Lemma 3.1 (Morris integral [3, (1.17)–(1.18)]). *Let $n \geq 2$, and let $a, b, \gamma \in \mathbb{C}$ satisfy*

$$\operatorname{Re}(a+b+1) > 0, \qquad \operatorname{Re}\gamma > -\min\left\{\frac{1}{n}, \frac{\operatorname{Re}(a+b+1)}{n-1}\right\}.$$

Then

$$\begin{aligned}\frac{1}{(2\pi)^n}\int_{[-\pi,\pi]^n}\prod_{r=1}^{n} e^{\frac{i}{2}(a-b)\theta_r}|1+e^{i\theta_r}|^{a+b}\prod_{1\leq r<s\leq n}|e^{i\theta_r}-e^{i\theta_s}|^{2\gamma}\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n \\ = \prod_{j=0}^{n-1}\frac{\Gamma(1+a+b+j\gamma)\Gamma(1+(j+1)\gamma)}{\Gamma(1+a+j\gamma)\Gamma(1+b+j\gamma)\Gamma(1+\gamma)}.\end{aligned}$$

For $n = 1$, the same identity holds under the single condition $\operatorname{Re}(a+b+1) > 0$.

Apply Lemma 3.1 with $n = N$, $a = b = k$, and $\gamma = 1$. The exponential factor is then 1. Moreover, translating every angle by π changes $|1+e^{i\theta_r}|$ into $|1-e^{i\theta_r}|$, while leaving all pairwise differences unchanged. Since $\Gamma(2) = 1$, the lemma gives

$$\begin{aligned}J_k(N) &= \frac{1}{N!}\prod_{j=0}^{N-1}\frac{\Gamma(1+2k+j)\Gamma(2+j)}{\Gamma(1+k+j)^2} = \frac{1}{N!}\prod_{r=1}^{N}\frac{\Gamma(2k+r)\Gamma(r+1)}{\Gamma(k+r)^2} \\ &= \prod_{r=1}^{N}\frac{\Gamma(r)\Gamma(2k+r)}{\Gamma(k+r)^2}.\end{aligned}$$

As $N \to +\infty$,

$$J_k(N) \sim \frac{G(1+k)^2}{G(1+2k)}N^{k^2},$$

where $G(z)$ is the Barnes G-function satisfying $G(1) = 1$ and $G(z+1) = \Gamma(z)G(z)$. For $n \in \mathbb{N}$,

$$G(n+1) = \prod_{j=0}^{n-1} j!.$$

Therefore, if used by itself, Keating–Snaith's random matrix calculation would predict

$$C_k = \frac{G(1+k)^2}{G(1+2k)}.$$

This vanilla prediction is incorrect because it omits the arithmetic factor a_k. Keating–Snaith corrected it by inserting a_k ad hoc, so the full conjecture is

$$I_k(T) \sim \frac{g_k a_k}{\Gamma(k^2+1)} (\log T)^{k^2}, \tag{3.3}$$

where

$$g_k = \Gamma(k^2+1) \frac{G(1+k)^2}{G(1+2k)}.$$

This is consistent with what we know at $k = 1, 2$. Since a_k has been inserted ad hoc, the vanilla random matrix model does not account for it; finding a model that does is a crucial motivation for the hybrid model below.

3.4 Hybrid Model

The Keating–Snaith model is not able to explain where the prime product a_k comes from. As an attempt to remedy this, Gonek–Hughes–Keating developed a hybrid model [6]. Roughly speaking,

$$\zeta(s) \approx \prod_{p \le X} (1 - p^{-s})^{-1} \times \prod_{\substack{\rho_n = \frac{1}{2} + i\gamma_n \\ |t-\gamma_n| \le 1/\log X}} \left[(s - \rho_n) e^{\gamma} \log X \right],$$

where $\gamma = 0.5772\ldots$ is Euler's constant. The first product is the Euler component P_X, and the second product is the Hadamard component Z_X.

Assuming these components behave independently in $[0, T]$ for a suitable $X = X(T)$, $I_k(T)$ is basically

$$I_k(T) \sim \frac{1}{T} \int_0^T |P_X|^{2k} \mathrm{d}t \cdot \frac{1}{T} \int_0^T |Z_X|^{2k} \mathrm{d}t. \tag{3.4}$$

Gonek–Hughes–Keating call (3.4) the splitting conjecture. Unconditionally, it was established that

$$\frac{1}{T} \int_0^T |P_X|^{2k} \mathrm{d}t \sim a_k (e^{\gamma} \log X)^{k^2}. \tag{3.5}$$

Z_X is basically $\det(I - ge^{-i\theta})$, so using ideas from the characteristic polynomial model, they conjectured that

$$\frac{1}{T} \int_0^T |Z_X|^{2k} \mathrm{d}t \sim \frac{G(1+k)^2}{G(1+2k)} \left(\frac{\log T}{e^{\gamma} \log X} \right)^{k^2}. \tag{3.6}$$

Plugging these into (3.4) recovers (3.3). The authors established (3.5) with $X \ll_{\varepsilon} (\log T)^{2-\varepsilon}$ and conjectured (3.4) and (3.6) hold in this same range. Heap (2023) [10] refined (3.5) to $X \ll_k (\log T)^{2+\delta_k}$ for some $\delta_k > 0$ and showed that the "$\asymp$" version of (3.4) is true in this larger range under RH.

3.5 Progress towards Moments of $|\zeta(\frac{1}{2} + it)|$

Asymptotics for $I_k(T)$ are only proved for $k = 1, 2$. The lower bounds for $I_k(T)$ have matched the conjectural magnitude:

$$I_k(T) \ge c_k (\log T)^{k^2}. \tag{3.7}$$

Progress on this lower bound is summarized below.

Work	Range of k
Ramachandra (1978, 1980) [23, 24]	Positive half-integers k
Heath-Brown (1981) [13]	Positive rational k
Radziwill–Soundararajan (2013) [22]	Real $k \ge 1$, with $c_k = e^{-30k^4}$
Heap–Soundararajan (2020) [12]	Real $0 < k \le 1$

The conjectural upper bound

$$I_k(T) \ll_k (\log T)^{k^2} \tag{3.8}$$

is true unconditionally for $0 < k \leq 2$ and true under RH for $k \geq 2$. The principal results leading to the current upper bounds are summarized below.

Work	Result
Hardy–Littlewood (1923) [8]	$I_k(T) \ll_{k,\varepsilon} T^{\varepsilon} \iff$ Lindelof hypothesis
Heath-Brown (1981) [13]	(3.8) unconditionally for $k = \frac{1}{n}, n \in \mathbb{N}$
Soundararajan (2009) [29]	$I_k(T) \ll_{k,\varepsilon} (\log T)^{k^2+\varepsilon}$ under RH
Harper (2013) [9]	(3.8) under RH
Bettin–Chandee–Radziwill (2017) [1]	(3.8) unconditionally for $k = 1 + \frac{1}{n}, n \in \mathbb{N}$
Heap–Radziwill–Soundararajan (2019) [11]	(3.8) unconditionally for real $0 < k \leq 2$

4 Generalization to L-functions

According to the Langlands conjectures, the most general L-functions we should study are those attached to automorphic representations of GL_N over number fields, and they should factor as products of *primitive L-functions* $L(s, \pi)$ attached to cuspidal automorphic representations π of GL_m over $\mathbb{Q}$, where m is called the *degree* of $L(s, \pi)$.

Examples include:

- $m = 1$: $\zeta(s)$ and $L(s, \chi)$ for χ primitive.
- $m = 2$: Hecke eigenforms and Maass forms.

$L(s, \pi)$ possesses an Euler product and analytic continuation. The central value $L(\frac{1}{2}, \pi)$ usually carries interesting arithmetical information, such as Tate–Shafarevich groups [2] and Fourier coefficients of cusp forms [32]. For a family $\mathcal{F}$ in which the central values are non-negative, we want to understand the moments

$$\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} L\left(\frac{1}{2}, f\right)^k. \tag{4.1}$$

According to Rudnick–Sarnak [25], the zeros for a single $L(s, \pi)$ are conjectured to have the same correlation statistics as the CUE, namely (2.6). However, the picture becomes very different when we look at $\{L(s, f)\}_{f \in \mathcal{F}}$.

4.1 Low-Lying Zeros and the Katz–Sarnak Philosophy

For a family $\mathcal{F}$, the most sensitive zeros are the *low-lying zeros*, namely zeros $\frac{1}{2} + i\gamma_f$ close to $s = \frac{1}{2}$. We assume RH for $\mathcal{F}$, so every γ_f is real. If the conductors in the family have size about C, the natural normalization is $\gamma_f \frac{\log C}{2\pi}$, so that the mean spacing is 1. Katz–Sarnak (1999) [16, 17] predicted that each reasonable $\mathcal{F}$ has a *symmetry type* $G(N)$, and the way to identify it is to compare the one-level density of these normalized zeros with the corresponding one-level density of eigenvalues near 1.

One tests the low-lying zeros with a Schwartz function φ. For $f \in \mathcal{F}$ with analytic conductor C_f, set

$$D(f; \varphi) = \sum_{\gamma_f} \varphi\left(\gamma_f \frac{\log C_f}{2\pi}\right),$$

where zeros are counted with multiplicity. For a finite subfamily $\mathcal{F}(Q) = \{f \in \mathcal{F} : C_f = Q\}$ or $\{f \in \mathcal{F} : C_f \leq Q\}$, the family statistic is the averaged one-level density

$$E(\mathcal{F}(Q); \varphi) = \frac{1}{|\mathcal{F}(Q)|} \sum_{f \in \mathcal{F}(Q)} D(f; \varphi).$$

The Katz–Sarnak prediction is that, as $Q \to \infty$, this average tends to an integral of φ against the density attached to the symmetry type. On the random matrix side, the candidate symmetry types are $U(N)$, $USp(2N)$, $SO(2N)$, $SO(2N+1)$, and $O(N)$. The full orthogonal group models families in which both signs of the functional equation occur, while the two special orthogonal groups model the corresponding fixed-sign subfamilies; the odd orthogonal case has a forced eigenvalue at 1, matching a negative sign. If $m = m(G(N))$ denotes the matrix size of the group $G(N)$, for example $m(U(N)) = N$, $m(USp(2N)) = 2N$, $m(O(N)) = N$, and $m(SO(2N+1)) = 2N+1$, then the random matrix local scaling is by $m/(2\pi)$. For $g \in G(N)$, order its eigen-angles as $0 \le \theta_1 \le \cdots \le \theta_m < 2\pi$ and extend them periodically by $\theta_{j+\ell m} = \theta_j + 2\pi\ell$. The matching one-level statistic is

$$D(g;\varphi) = \sum_{j\in\mathbb{Z}} \varphi\left(\frac{m\theta_j}{2\pi}\right),$$

and its Haar average is

$$E(G(N);\varphi) = \int_{g\in G(N)} D(g;\varphi)\mathrm{d}_{\mathrm{Haar}}g.$$

The density comparison says that

$$\lim_{N\to\infty} E(G(N);\varphi) = \int_{\mathbb{R}} \varphi(x)D_G(x)\mathrm{d}x.$$

Katz–Sarnak [16, Theorem A.D.12.6] computed the large-N one-level scaling densities near 1 for the classical compact groups. For compactly supported test functions, the limiting densities have the form

$$\begin{aligned} D_U(x) &= 1, \\ D_{USp}(x) &= 1 - \frac{\sin 2\pi x}{2\pi x}, \\ D_O(x) &= 1 + \frac{1}{2}\delta(x), \\ D_{SO(\mathrm{even})}(x) &= 1 + \frac{\sin 2\pi x}{2\pi x}, \\ D_{SO(\mathrm{odd})}(x) &= \delta(x) + 1 - \frac{\sin 2\pi x}{2\pi x}. \end{aligned}$$

Thus unitary symmetry has no special behavior at the central point, symplectic symmetry repels zeros from the center, and odd orthogonal symmetry includes a forced central zero. These distinctions are invisible in the high-zero CUE statistics of a single L-function, but they are visible when the family is averaged.

As with Montgomery's conjecture, known results in this subject tend to require some support condition on the types of test functions involved, for example, Iwaniec–Luo–Sarnak (2000) [15] and Fouvry–Iwaniec (2003) [4]. However, Katz–Sarnak (1999) [16, 17] carried out their philosophy successfully in function fields, where more is known, such as RH. Based on observations in function fields and partial results, we have the following example classifications:

Table 1: Examples of Katz–Sarnak symmetry types

Family $\mathcal{F}$	Limit	Symmetry type
{primitive χ mod p}	$p \to \infty$	Unitary
$\{\chi_d = (\frac{d}{\cdot}) : d \in \mathcal{D}(X)\}$	$X \to \infty$	Symplectic
{weight k Hecke eigenforms on $SL_2(\mathbb{Z})$, $k \ge 2$ even}	$k \to \infty$	Orthogonal

Here $\mathcal{D}(X)$ is the set of fundamental discriminants d with $|d| \le X$. For the weight k Hecke eigenforms in the orthogonal family, the sign of the functional equation is $(-1)^{k/2}$. Thus, $k \equiv 0 \pmod 4$ corresponds to $SO(2N)$, while $k \equiv 2 \pmod 4$ corresponds to $SO(2N+1)$; combining both congruence classes gives $O(N)$ symmetry.

For a self-dual family with non-negative central values, moments are modeled by the corresponding characteristic-polynomial integral

$$\int_{g\in G(N)} \det(I-g)^k \mathrm{d}_{\mathrm{Haar}}g, \tag{4.2}$$

where $G(N)$ is the compact group attached to the family.

For unitary families, whose central values are generally complex, one instead considers absolute moments or mixed moments and uses the corresponding moments of complex characteristic polynomials.

4.2 Moments of Central Values after Katz–Sarnak

As in the case of $\zeta(s)$, the conjectural asymptotic for the family moments in (4.1) has the form

$$\frac{1}{|\mathcal{F}(Q)|}\sum_{f\in\mathcal{F}(Q)} L\left(\frac{1}{2},f\right)^k \sim \text{RMT coefficient} \times \text{arithmetic factors} \times \text{power of logs}.$$

The RMT coefficient comes from the Katz–Sarnak symmetry type. Arithmetic factors come from the Euler product for $L(s,f)$. The exact power of logs depends on the statistics of $\log\det(I-g)$.

In the rest of this subsection, we illustrate this recipe only for the quadratic Dirichlet family $\mathcal{F}(X) = \{\chi_d : d \in \mathcal{D}(X)\}$, whose Katz–Sarnak symmetry type is symplectic.

For this family, the random matrix model is $USp(2N)$. Keating–Snaith [18] computed the corresponding statistics. In this model, $\log\det(I-g)$ is approximately normal with mean $\frac{1}{2}\log N$ and variance $\log N$. The mean spacing of low-lying zeros is $\frac{2\pi}{\log X}$. Equating this with $\frac{2\pi}{2N}$ gives $N \approx \frac{1}{2}\log X$.

Although the analogue of Selberg's CLT for this family remains conjectural, the following heuristic is due to Soundararajan [30, §3]:

$$\begin{aligned}\log L\left(\frac{1}{2},\chi_d\right) &\approx \sum_{n\le x}\frac{\Lambda(n)}{\sqrt{n}\log n}\chi_d(n)\\ &\approx \sum_{p\le x}\frac{\chi_d(p)}{\sqrt{p}} + \frac{1}{2}\sum_{\substack{p\le\sqrt{x}\\ p\nmid d}}\frac{1}{p} \approx \sum_{p\le x}\frac{\chi_d(p)}{\sqrt{p}} + \frac{1}{2}\log\log X, \qquad x = X^c,\end{aligned}$$

The moments of the prime sum satisfy

$$\frac{1}{|\mathcal{D}(X)|}\sum_{d\in\mathcal{D}(X)}\left(\sum_{p\le x}\frac{\chi_d(p)}{\sqrt{p}}\right)^k = \sum_{p_1,\dots,p_k\le x}\frac{1}{\sqrt{p_1\cdots p_k}}\cdot\frac{1}{|\mathcal{D}(X)|}\sum_{d\in\mathcal{D}(X)}\chi_d(p_1\cdots p_k). \tag{4.3}$$

The inner sum

$$\frac{1}{|\mathcal{D}(X)|}\sum_{d\in\mathcal{D}(X)}\chi_d(n)$$

is essentially an indicator for squares, so (4.3) goes to zero for odd k. For even k, there are exactly

$$(k-1)(k-3)\cdots 1 = \frac{k!}{2^{k/2}(k/2)!}$$

ways to group k objects into $k/2$ pairs, so (4.3) is

$$\approx \frac{k!}{2^{k/2}(k/2)!}\left(\sum_{p\le x}\frac{1}{p}\right)^{k/2} \sim \frac{k!}{2^{k/2}(k/2)!}(\log\log X)^{k/2}.$$

Therefore, $\log L(\frac{1}{2},\chi_d)$ may be regarded as normal with mean $\frac{1}{2}\log\log X$ and variance $\log\log X$. Thus, we can model $L(\frac{1}{2},\chi_d)$ via $\det(I-g)$.

The RMT calculation gives

$$\int_{USp(2N)}\det(I-g)^k \mathrm{d}_{\mathrm{Haar}}g \sim 2^{k^2/2}\frac{G(1+k)\sqrt{\Gamma(1+k)}}{\sqrt{G(1+2k)\Gamma(1+2k)}}N^{k(k+1)/2}.$$

The arithmetic factor is

$$a_k = \prod_p\left(1-\frac{1}{p}\right)^{k(k+1)/2}\left(\frac{p}{2(p+1)}\left[\left(1+\frac{1}{\sqrt{p}}\right)^{-k} + \left(1-\frac{1}{\sqrt{p}}\right)^{-k}\right] + \frac{1}{p+1}\right).$$

Therefore, the Keating–Snaith conjecture for $L(s,\chi_d)$ [18] is

$$\frac{1}{|\mathcal{D}(X)|}\sum_{d\in\mathcal{D}(X)} L\left(\frac{1}{2},\chi_d\right)^k \sim 2^{k^2/2}\frac{G(1+k)\sqrt{\Gamma(1+k)}}{\sqrt{G(1+2k)\Gamma(1+2k)}}\cdot a_k(\log\sqrt{X})^{k(k+1)/2}.$$

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