Additive Divisor Problem

In memoriam Professor Ju. V. Linnik

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Introduction

The *additive divisor problem** refers to estimating sums of the types:

$$D^{+}_{k_1,k_2}(N,h) = \sum_{n\le N} \tau_{k_1}(n)\tau_{k_2}(n+h), \quad h \in \mathbb{Z}, \tag{0.1}$$

$$D^{-}_{k_1,k_2}(N) = \sum_{n<N} \tau_{k_1}(n)\tau_{k_2}(N-n), \tag{0.2}$$

in which τ_k is the k-fold divisor function.

Alternatively, one can interpret these sums as the number of solutions to Diophantine equations with positive integers restricted in certain regions.

For instance, (0.1) counts the number of solutions to

$$x_1x_2\cdots x_{k_1} - y_1y_2\cdots y_{k_2} = h \quad s.t. \quad y_1\cdots y_{k_1} \le N,$$

and (0.2) counts that of

$$x_1x_2\cdots x_{k_1} + y_1y_2\cdots y_{k_2} = N.$$

If we treat $x_1x_2\cdots x_{k_1}$ and $y_1y_2\cdots y_{k_2}$ as elements in sets weighted by τ_{k_1} and τ_{k_2}, then both (0.1) and (0.2) can be regarded as instances of *binary additive problems*, which concerns the solutions (α,β) to the equation

$$\alpha + \beta = N \tag{0.3}$$

when α,β range over interesting sets of integers. For instance, if we range over primes, then (0.3) corresponds to the Goldbach problem. Due to our limited understanding of primes, many binary additive problems were solved only under the assumption of strong hypotheses. For instance, Heath-Brown [8] proved that the twin primes problem ($N = 2$, $\alpha =$ primes, and $\beta =$ negative primes) is only known to be true under the existence of Landau–Siegel zeros.

In the 1960s, Ju. V. Linnik and his school developed the *dispersion method* and achieved complete solutions to a number of binary additive problems that were previously true only conditionally. For instance, in 1923, Hardy and Littlewood [7] considered the equation

$$p + x^2 + y^2 = N$$

and conjectured that the number $H(N)$ of solutions p, x, y is

$$H(N) \sim \frac{\pi N}{\log N} \prod_{2<p|n} \frac{(p-1)(p-(-1)^{\frac{p-1}{2}})}{p^2-p+(-1)^{\frac{p-1}{2}}} \prod_{p>2}\left(1+\frac{(-1)^{\frac{p-1}{2}}}{p(p-1)}\right). \tag{0.4}$$

*It is also known as the additive correlation and the shifted convolution of divisor functions in literature.

In 1957, Hooley [10] showed that (0.4) holds under GRH. In 1960, Linnik [15] removed this requirement.

The *Titchmarsh divisor problem* asks for solutions to the equation

$$p - xy = 1,$$

where x, y are integers and p is a prime $\leq N$. In 1930, Titchmarsh [22] showed that the number $T(N)$ of solutions p, x, y to this equation is

$$T(N) \sim \frac{\zeta(2)\zeta(3)}{\zeta(6)} N \tag{0.5}$$

under the truth of the Generalized Riemann Hypothesis (GRH).

In regard to the additive divisor problems (0.1) and (0.2), asymptotic formulas were only obtained for small tuples (k_1, k_2) before Linnik. In 1927, Ingham [12] studied the case $(2,2)$ and obtained

$$D^+_{2,2}(N,h) \sim \frac{6}{\pi^2}\sigma_{-1}(a)N(\log N)^2, \tag{0.6}$$

$$D^-_{2,2}(N,h) \sim \frac{6}{\pi^2}\sigma_1(N)(\log N)^2, \tag{0.7}$$

where $\sigma_s(n) = \sum_{d|n} d^s$. In 1957, Hooley [9] obtained formulas of (0.1) and (0.2) for the cases $(2,3)$ and $(3,2)$ using the Weil bound for Kloosterman sums. For the case $h = 1$, the formulas for $D^+_{k_1,k_2}(N,h)$ reduce to

$$D^+_{2,3}(N,1) \sim D^+_{3,2}(N,1) \sim \frac{1}{2}\prod_p \left(1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^2\right) N(\log N)^2. \tag{0.8}$$

In 1958, Linnik [14] obtained asymptotic formulas for (0.1) and (0.2) in the case of $(2,k)$ and $(k,2)$ for all $k \geq 3$. For (0.1) with $h = 1$, the formulas reduce to

$$D^+_{2,k}(N,1) \sim D^+_{k,2}(N,1) \sim \frac{1}{(k-1)!}\prod_p \left(1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^{k-1}\right) N(\log N)^k. \tag{0.9}$$

Linnik also claimed that his formula can be developed into an asymptotic series with a power saving error term. This was not validated until subsequent works of Motohashi [20], Fouvry and Tenenbaum [5], and Topacogullari [23]. It is also worth mentioning that conjectured asymptotics for general $D^+_{k_1,k_2}(N,h)$ are known, and they were confirmed by Matomäki, Radziwiłł, and Tao [17] [18] in "almost all" sense. That is, when $H > 0$ grows with N at a proper rate, the conjectured asymptotics for $D^+_{k_1,k_2}(N,h)$ for all but an $o(H)$ amount of $h \in [-H, H]$.

In 1965, E. Bombieri [1] and A. I. Vinogradov [24] independently established a strong equidistribution result concerning primes in arithmetic progressions.

In 1966, using the Bombieri–Vinogradov theorem, Elliott and Halberstam [4] obtained unconditional solutions to the Titchmarsh divisor problem and the Hardy-Littlewood problem. As their methods were considerably simpler, literature discussing these problems [11] [2] tend to elaborate on the Elliott–Halberstam approach and only citing Linnik for providing the first solutions.

As a result, Linnik's solutions are less documented. The author believed exploring the connection of Linnik's methods with modern ideas may shed new light on binary additive problems that are still unsolved to this date.

Most of Linnik's publications on the binary additive problems were in the form of announcements or technical fragments. In 1963, he assembled his works into the monograph *The dispersion method in binary additive problems* [16]. Due to the abundance of technical details, the derivations were written in a compressed manner, making it difficult to follow. As a result, the author felt obliged to give a more reader friendly treatment of Linnik's method.

A key feature in Linnik's approach to the Hardy–Littlewood problem and the Titchmarsh divisor problem is the combinatorial identity:

$$\log \zeta(s) = \log(1 + \zeta(s) - 1) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} (\zeta(s) - 1)^k,$$

allowing one to convert problems concerning primes to problems concerning certain divisor functions. This means both of these problems can be reduced to a situation similar to that of the additive divisor problem, which is why we chose it as the primary subject of investigation.

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Notation

p denotes a prime number.

$s = \sigma + it$ refers to a complex number with real part σ and imaginary part t.

$\rho = \beta + i\gamma$ refers to a zero of the Riemann zeta function $\zeta(s)$ in the critical strip with real part β and imaginary part γ.

$n \equiv a(q)$ means $n \equiv a \pmod{q}$.

$\pi(x; q, a) = \sum_{\substack{p \leq x \\ n \equiv a(q)}} 1$ and $\pi(x) = \pi(x; 1, 1)$.

$\mathrm{li}(x)$ is the logarithmic integral defined by the principal value integral

$$\lim_{\varepsilon \to 0^+} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{x} \right) \frac{\mathrm{d}u}{\log u}.$$

$\Lambda(n)$ is the von Mangoldt function equal to $\log p$ if $n = p^k$ and zero otherwise.

$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a(q)}} \Lambda(n)$ and $\psi(x) = \psi(x; 1, 1)$.

$q \sim Q$ refers to the range $Q < q \leq 2Q$.

$\mathcal{L} = \log N$.

1 Outline

In this document, we will prove

Theorem 1.1. *Let*

$$D_k(N,a) = \sum_{n\le N} \tau_k(n)\tau(n+a). \tag{1.1}$$

Then for $k \ge 3$ fixed and $N \to +\infty$,

$$D_k(N,1) = \frac{1}{(k-1)!}\prod_p \left(1-\frac{1}{p}+\frac{1}{p}\left(1-\frac{1}{p}\right)^{k-1}\right) NF_k(\log N) + O(N), \tag{1.2}$$

where $F_k(x)$ is a monic polynomial of degree k.

A key feature in Linnik's approach is replacing $D_k(N,1)$ with an average over some other $D_k(N,a)$. Specifically, let K_1 be large and fixed and $P = \lfloor N\mathcal{L}^{-K_1}\rfloor$. Then Linnik showed that

$$D_k(N,1) = \frac{\log P}{P}\sum_{p\sim P} D_k(N,p) + O(N). \tag{1.3}$$

Because of this averaging phenomenon, the numbers in the range of summation are known as the *coherent numbers*.

The sum in the right hand side of (1.3) counts the number of solutions to the Diophantine equation

$$xy - x_1x_2\cdots x_k = p, \quad xy \le N, p \sim P. \tag{1.4}$$

This is a ternary additive problem that can be treated using the circle method: the number of solutions is precisely

$$\int_0^1 T_2(-\alpha;N)T_k(\alpha;N)\sum_{p\sim P} e(p\alpha)\mathrm{d}\alpha,$$

where

$$T_k(\alpha;x) = \sum_{n\le x}\tau_k(n)e(n\alpha). \tag{1.5}$$

1.1 The averaging phenomenon

The divisor function $\tau_k(n)$ can take values as large as $n^{C/\log\log n}$, which is inconvenient for applying ℓ^∞-norms in certain estimations. Fortunately, moment estimates for $\tau_k(n)$ suggest that the main contribution to $D_k(N,a)$ comes from situations where $\tau_k(n) \le \mathcal{L}^A$. By some sieve-theoretic arguments, one can further restrict to the situations where n possesses a prime factor in some interval $(Y,Z]$ with

$$Y = \exp(\sqrt{\mathcal{L}}), \quad Z = N^{u_0},$$

where $u_0 > 0$ is small and fixed (in fact, one can take $u_0 = \frac{1}{17}$).

Proposition 1.1. *Let $a = 1$ or a prime $\sim P$ and $n = n_1 n_2$, where n_1 consists of all prime factors of n in $(Y, Z]$. Then for $A \gg_k 1$ fixed,*

$$D_k(N,a) = \sum_{\substack{n \le N \\ \tau_k(n) \le \mathcal{L}^A \\ n_1 > 1}} \tau(n+a)\tau_k(n) + O(N).$$

In this range, n can be written as pm with $p|Q$ and $m \le N/p$. If n_1 is composite, then this representation is not unique. However, a further technical calculation allows us to focus only on the cases where n_1 is squarefree. When n_1 is a product of ℓ distinct primes, there are exactly ℓ ways to factor n into pm with $p \in (Y, Z]$. By the multiplicative property of $\tau_k(n)$, we have $\tau_k(n) = k\tau_k(m)$, allowing us to obtain

$$\sum_{\substack{n \le N \\ \tau_k(n) \le T \\ n_1 > 1, \mu(n_1) \ne 0, \omega(n_1) = \ell}} \tau(n+a)\tau_k(n) = \frac{k}{\ell} \sum_{Y < p \le Z} \sum_{m \le N/p} a_{m,\ell} \tau(pm+a),$$

where $a_{m,\ell} = \tau_k(m)$ if

$$\tau_k(m) \le \mathcal{L}^A/k, \quad m_1 \text{ squarefree}, \quad \omega(m_1) = \ell - 1,$$

and $a_{m,\ell} = 0$ otherwise. Here, m_1 is defined in a manner similar to n_1.

Combining these observations, $D_k(N,a)$ becomes

Proposition 1.2. *Let $a = 1$ or a prime $\sim P$. Define*

$$D_{k,\ell}(N,a) = \sum_{Y < p \le Z} \sum_{m \le N/p} a_{m,\ell} \tau(pm+a).$$

Then we have

$$D_k(N,a) = \sum_{\ell \le u_0 \sqrt{\mathcal{L}}} \frac{k}{\ell} D_{k,\ell}(N,a) + O(N).$$

For each $D_{k,\ell}(N,a)$, the ranges of p and m are subdivided into $\ll \mathcal{L}^{O(1)}$ small boxes, so the task is reduced to handling

$$S_{k,\ell}(N,a;D,M) = \sum_{M < m \le \eta_2 M} a_{m,\ell} \sum_{D < p \le \eta_1 D} \tau(pm+a), \tag{1.6}$$

where $\eta_j = 1 + \mathcal{L}^{-H_j}$ and H_1, H_2 are large constants, and

$$Y \le D \le Z, \quad M \le N/D. \tag{1.7}$$

By Cauchy–Schwarz, one has

$$|S_{k,\ell}(N,1;D,M) - S_{k,\ell}(N,a;D,M)| \le \mathcal{L}^{A - \frac{1}{2}H_2} M^{\frac{1}{2}} V(N,a;D,M)^{\frac{1}{2}}, \tag{1.8}$$

where $V(N,a;D,M)$ is the *dispersion*

$$V(N,a;D,M)=\sum_{M<m\le\eta_2 M}\left(\sum_{D<p\le\eta_1 D}\tau(pm+1)-\sum_{D<p\le\eta_2 D}\tau(pm+a)\right)^2. \tag{1.9}$$

By opening the square, one has

$$V(N,a;D,M)=\sum_{D<p_1,p_2\le\eta_1 D}(W(1,1;p_1,p_2)-2W(1,a;p_1,p_2)+W(a,a;p_1,p_2)), \tag{1.10}$$

in which (the dependence on N,D,M is suppressed)

$$W(a_1,a_2;p_1,p_2)=\sum_{M<m\le\eta_2 M}\tau(p_2m+a_1)\tau(p_1m+a_2). \tag{1.11}$$

$W(a_1,a_2;p_1,p_2)$ counts the number of solutions to a certain Diophantine equation in a complicated region. By a series of change of variables and eliminations, Linnik is able to reduce $W(a_1,a_2;p_1,p_2)$ to a form that allows the method of exponential sums to be applied, thereby obtaining an asymptotic count of the number of solutions, yielding

Proposition 1.3. *Let D,M be as in* (1.7). *If a_j is either 1 or a prime $\sim P$, then*

$$\sum_{D<p_1,p_2\le\eta_1 D}W(a_1,a_2;p_1,p_2)=\Sigma+O(D^2M\mathcal{L}^{-\frac{1}{2}H_3}+D^4M^{\frac{3}{4}}N^{\varepsilon}+D^2N^{\frac{1}{2}+\varepsilon}),$$

where

$$\Sigma=4\left(\int_D^{\eta_1 D}\frac{\mathrm{d}u}{\log u}\right)^2 M\mathcal{L}^{-H_2}\sum_{\delta\le\mathcal{L}^{H_3}}\frac{1}{\delta\varphi(\delta)}\sum_{\substack{x\le\sqrt{DM}/\delta\\(x,z)=1}}\sum_{z\le\sqrt{DM}/\delta}\frac{1}{xz},$$

and $H_3\gg H_1,H_2$ is a large constant.

Observing that Σ does not depend on the specific choice of a_1,a_2, plugging Proposition 1.3 into (1.10) gives

$$V(N,a;D,M)\ll D^2M\mathcal{L}^{-\frac{1}{2}H_3}+D^4M^{\frac{3}{4}}N^{\varepsilon}+D^2N^{\frac{1}{2}+\varepsilon}. \tag{1.12}$$

provided that a is a prime $\sim P$. Plugging this into (1.8), we obtain

$$\begin{aligned}|S_{k,\ell}(N,1;D,M)-S_{k,\ell}(N,a;D,M)|&\\ \ll DM(\log N)^{-\frac{1}{3}H_3}+D^2M^{\frac{7}{8}}N^{\varepsilon}+DM^{\frac{1}{2}}N^{\frac{1}{4}+\varepsilon}.&\end{aligned}$$

Finally, this estimate is used to deduce the averaging phenomenon:

Proposition 1.4. *If a is a prime $\sim P$, then*

$$D_k(N,1) - D_k(N,a) = O(N).$$

Thus, the task of estimating $D_k(N,a)$ has now been reduced to the solving the ternary additive problem (1.4).

Remark. The error bound $O(N)$ in Theorem 1.1 and Proposition 1.4 comes from our wasteful treatments in error terms emerging in the proof of Proposition 1.1. Using Linnik's large sieve inequality, Motohashi [20] was able to exploit the cancellation of these extra terms in $D(N,1) - D(N,a)$ and thus capable of refining Theorem 1.1 to an asymptotic expansion with error $O(N(\log \mathcal{L})^c \mathcal{L}^{-1})$.

1.2 Solution to the ternary problem

By (1.4), we have

$$D_k(N,1) = \underbrace{\frac{1}{P} \sum_{p \sim P} (\log p) D_k(N,p)}_{X_k(N)} + O(N), \tag{1.13}$$

which is a weighted version of (1.3) that is easier to work with.

Define

$$S(\alpha) = \sum_{p \sim P} (\log p) e(-p\alpha), \tag{1.14}$$

so we have

$$X_k(N) = \int_0^1 T_2(-\alpha; N) T_k(\alpha; N) S(\alpha) \mathrm{d}\alpha. \tag{1.15}$$

According to the Hardy–Littlewood–Vinogradov framework, we divide $[0,1]$ into major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m} = [0,1] \setminus \mathfrak{M}$. The major arcs are defined as follows:

$$\mathfrak{M} = \bigcup_{\substack{1 \le a \le q \le \mathcal{L}^{K_2} \\ (a,q)=1}} \mathfrak{M}(a/q), \quad \mathfrak{M}(a/q) = \left[\frac{a}{q} - \frac{\mathcal{L}^{K_2}}{N}, \frac{a}{q} + \frac{\mathcal{L}^{K_2}}{N} \right], \tag{1.16}$$

in which the subintervals are mutually disjoint because for reduced fractions $a/q \ne a'/q'$ with $q, q' \le \mathcal{L}^{K_2}$,

$$\left| \frac{a}{q} - \frac{a'}{q'} \right| = \frac{|aq' - a'q|}{qq'} \ge \frac{1}{qq'} \ge \frac{1}{\mathcal{L}^{2K_2}} > \frac{2\mathcal{L}^{K_2}}{N}.$$

To determine the contribution to (1.15) from major arcs, we require asymptotics for the exponential sums:

Proposition 1.5 ([3, p. 147]). *There exists some $c = c(K_2) > 0$ for which when $q \le \mathcal{L}^{K_2}$ and $(a,q) = 1$, one has*

$$S\left(\frac{a}{q} + \beta\right) = \frac{\mu(q)}{\varphi(q)} S_0(\beta) + O((1 + |\beta| N) e^{-c\sqrt{\mathcal{L}}}),$$

where

$$S_0(\beta) = \sum_{n \sim P} e(n\beta). \tag{1.17}$$

Observe that the main term of $S(\alpha)$ involves $\mu(q)$, so we only need to estimate $T_k(\alpha; N)$ when α is near some a/q with q squarefree:

Proposition 1.6. *When $q \le \mathcal{L}^{K_2}$ is squarefree and $(a,q) = 1$, one has*

$$T_k\left(\frac{a}{q} + \beta; N\right) = \frac{1}{\varphi(q)} U_k(\beta; q) + O((1 + |\beta| N) N^{1 - \frac{1}{k+1} + \varepsilon}),$$

where

$$U_k(\beta; q) = \sum_{n \le N} e(n\beta) \Gamma_k(n; q), \tag{1.18}$$

$$\Gamma_k(n; q) = \sum_{j=0}^{k-1} \theta_k^{(j)}(q)\, x(\log x)^{k-1-j}\Big|_{n-1}^{n}, \quad (f(x)|_a^b := f(b) - f(a)), \tag{1.19}$$

and $\theta_k^j(q)$ are some numbers such that

$$\theta_k^{(0)}(q) = \frac{\varphi(q)}{(k-1)!} \prod_{p|q} \left(1 - \left(1 - \frac{1}{p}\right)^{k-1}\right), \quad \theta_k^{(j)}(q) \ll q^{\varepsilon}. \tag{1.20}$$

As for the minor arcs, note that

$$\left|\int_{\mathfrak{m}} T_2(-\alpha; N) T_k(\alpha; N) S(\alpha) \mathrm{d}\alpha\right| \le \sup_{\alpha \in \mathfrak{m}} |S(\alpha)| \int_0^1 |T_2(-\alpha; N) T_k(\alpha; N)| \mathrm{d}\alpha$$
$$\le \frac{1}{2} \sup_{\alpha \in \mathfrak{m}} |S(\alpha)| \int_0^1 (|T_2(-\alpha; N)|^2 + |T_k(\alpha; N)|^2) \mathrm{d}\alpha,$$

so it suffices to find an upper bound for $S(\alpha)$, which can be deduced from the following classical result originally due to I. M. Vinogradov:

Proposition 1.7 ([3, p. 143]). *If $(a,q) = 1$ and $\alpha \in \mathbb{R}$ is such that*

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2},$$

then one has

$$S(\alpha) \ll (Pq^{-\frac{1}{2}} + P^{\frac{4}{5}} + P^{\frac{1}{2}} q^{\frac{1}{2}}) \mathcal{L}^4.$$

Collecting all these estimates, we arrive at an asymptotic formula for $X_k(N)$:

Proposition 1.8. *Let $K > 0$ be fixed. Then there exists constants $\{\xi_k(\ell)\}_{0\le\ell\le k}$ such that*

$$X_k(N) = PN \sum_{\ell=0}^{k} \xi_k(\ell)(\log N)^{k-\ell} + O(PN(\log N)^{-K}).$$

In particular,

$$\xi_k(0) = \frac{1}{(k-1)!} \prod_p \left(1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^{k-1}\right).$$

Combining Proposition 1.8 with (1.13), we deduce Theorem 1.1.

2 The averaging phenomenon

In this section, we give proofs to propositions stated in §1.1. Specifically, Proposition 1.1 and Proposition 1.2 are derived in §2.1, and Proposition 1.3 is proved in §2.5. Finally, Proposition 1.4 is proved in §2.6.

2.1 Elementary reductions

Recall that

$$D_k(N,a) = \sum_{n\le N} \tau_k(n)\tau(n+a).$$

To estimate the contributions to $D_k(N,a)$ from large values of τ_k, we need an estimate on the moments:

Lemma 2.1 (P. Shiu [21]). *Let $k\ge 1, \ell\ge 2$ be integers and $0<\alpha,\beta<\frac{1}{2}$. If*

$$1\le q\le y^{1-\beta}, \quad x^{\beta}<y\le x,$$

then as $x\to+\infty$, one has

$$\sum_{\substack{x-y<n\le x\\ n\equiv a(q)}} \tau_k^{\ell}(n) \ll \frac{y}{q}\left(\frac{\varphi(q)}{q}\log x\right)^{k^{\ell}-1},$$

where the $\ll$-constant depends only on k,ℓ,α,β.

We also require estimates on rough numbers and smooth numbers:

Lemma 2.2 (Brun–Titchmarsh). *Let $0<u_0<\frac{1}{4}$ be fixed. Then for $q\le x^{\frac{1}{2}}$ and $(a,q)=1$, one has*

$$\#\{n\le x : n\equiv a \pmod q, p|n \Rightarrow p>x^{u_0}\} \ll \frac{x}{\varphi(q)\log x},$$

where the $\ll$-constant depends only on u_0.

Proof. By Selberg's sieve [6, Theorem 3.6], the left hand side is

$$\ll \frac{x}{\varphi(q)\log x} + x^{2u_0}.$$

Note that $2u_0<\frac{1}{2}$, so

$$x^{2u_0} \ll \frac{x^{\frac{1}{2}}}{\log x} \ll \frac{x}{q\log x} \le \frac{x}{\varphi(q)\log x}.$$

□

Lemma 2.3 (Rankin)**.** *We have*

$$\#\{n \le x : p|n \Rightarrow p \le \exp(\sqrt{\log x})\} \ll x \exp\left(-\frac{1}{2}\sqrt{\log x}\right). \tag{2.1}$$

Proof. Write $z = \exp(\sqrt{\log x})$ and set $\delta \in (0,1)$. Then the left-hand side is

$$\le \sum_{\substack{n \\ p|n \Rightarrow p \le z}} \left(\frac{x}{n}\right)^{1-\delta} = x^{1-\delta} \prod_{p \le z} \left(1 + \frac{1}{p^{1-\delta}}\right) \le x^{1-\delta} \exp\left(\sum_{p \le z} \frac{p^\delta}{p}\right).$$

By $p^{-\delta} \ge 1 - \delta \log p$, we have $p^\delta \le 1 + \delta(\log p)p^\delta$, so

$$\begin{aligned}\sum_{p \le z} \frac{p^\delta}{p} &\le \sum_{p \le z} \frac{1}{p} + \delta z^\delta \sum_{p \le z} \frac{\log p}{p} \\ &\le \log\log z + O(1) + O(\delta(\log z) z^\delta).\end{aligned}$$

Setting $\delta = (\log z)^{-1}$, we see that this is $\le \log\log z + O(1)$. Therefore, the left hand side of (2.1) is

$$\ll x(\log z)\exp\left(-\frac{\log x}{\log z}\right) \ll x \exp\left(-\frac{1}{2}\sqrt{\log x}\right).$$

□

To handle sums over bounds coming from Lemma 2.2, we also need an estimate on the Euler φ-function:

Lemma 2.4. *We have*

$$\sum_{n \le x} \frac{1}{\varphi(n)} = O(\log x). \tag{2.2}$$

Proof.

$$\sum_{n \le x} \frac{1}{\phi(n)} \le \sum_{\substack{n \\ p|n \Rightarrow p \le x}} \frac{1}{\varphi(n)} = \prod_{p \le z} \left(1 + \sum_{r \ge 1} \frac{1}{\varphi(p^r)}\right). \tag{2.3}$$

Observe that

$$\sum_{r \ge 1} \frac{1}{\varphi(p^r)} = \sum_{r \ge 1} \frac{1}{p^r(1-p^{-1})} = \frac{p^{-1}}{(1-p^{-1})^2} = \frac{1}{p} + O\left(\frac{1}{p^2}\right),$$

so the right-hand side of (2.3) is

$$\le \exp\left(\sum_{p \le x} \frac{1}{p} + O(1)\right) \ll \log x.$$

□

Proof of Proposition 1.1. By Lemma 2.1, the contribution of terms with $\tau_k(n) > \mathcal{L}^A$ is

$$\sum_{\substack{n\le N\\ \tau_k(n)>\mathcal{L}^A}} \tau(n+a)\tau_k(n) \le \mathcal{L}^{-A}\sum_{n\le N}\tau(n+a)\tau_k^2(n)$$

$$\le \mathcal{L}^{-A}\left(\sum_{a<m\le N+a}\tau^2(m)\right)^{\frac12}\left(\sum_{n\le N}\tau_k^4(n)\right)^{\frac12}$$

$$\ll N\mathcal{L}^{\frac12(2^2-1+k^4-1)-A} = N\mathcal{L}^{\frac12 k^4+1-A} \ll N$$

provided that $A \ge \frac12 k^4+1$.

When the prime factors of $n \le N$ are $> Z = N^{u_0}$, n is a product of at most $r = \lceil u_0^{-1}\rceil - 1$ different primes, each with multiplicity $\le r$, so

$$\tau_k(n) \le \binom{r+k-1}{k-1}^r = O(1).$$

Consequently, among the terms satisfying $\tau_k(n) \le \mathcal{L}^A$, the contribution of those without prime factors $\le Z = N^{u_0}$ $(0 < u_0 < \frac14)$ is

$$\sum_{\substack{n\le N\\ \tau_k(n)\le\mathcal{L}^A\\ p|n\Rightarrow p>Z}} \tau(n+a)\tau_k(n) \ll \sum_{\substack{n\le N\\ p|n\Rightarrow p>Z}}\tau(n+a) = \sum_{\substack{n\le N\\ p|n\Rightarrow p>Z}}\left(\mathbf{1}_\square(n+a) + 2\sum_{\substack{d|(n+a)\\ d<\sqrt{n+a}}}1\right)$$

$$\ll N^{\frac12} + \sum_{d\le\sqrt{N+a}}\ \sum_{\substack{n\le N\\ n\equiv -a(d)\\ p|n\Rightarrow p>Z}} 1,$$

where $\mathbf{1}_\square$ is the characteristic function for perfect squares.

By our assumptions, a is free of prime divisors $\le \sqrt{2N}$, so $(a,d)=1$. Thus, it follows from Lemma 2.2 and Lemma 2.4 that the right-hand side of $\ll$ is

$$\ll N^{\frac12} + N\mathcal{L}^{-1}\sum_{d\le\sqrt{2N}}\frac{1}{\varphi(d)} \ll N.$$

Among the terms satisfying $\tau_k(n) \le \mathcal{L}^A$, the contribution of those without prime factors $> Y = \exp(\sqrt{\mathcal{L}})$ is

$$\sum_{\substack{n\le N\\ \tau_k(n)\le\mathcal{L}^A\\ p|n\Rightarrow p\le Y}} \tau_k(n)\tau(n+a) \le \mathcal{L}^A\sum_{\substack{n\le N\\ p|n\Rightarrow p\le Y}}\tau(n+a)$$

$$\le \mathcal{L}^A\left(\sum_{a<m\le N+a}\tau^2(m)\right)^{\frac12}\sqrt{\#\{n\le N : p|n\Rightarrow p\le Z\}}.$$

Hence, it follows from Lemma 2.1 and Lemma 2.3 that this quantity is

$$\ll \mathcal{L}^A (N\mathcal{L}^{2^2-1})^{\frac{1}{2}} (Ne^{-\frac{1}{2}\sqrt{\mathcal{L}}})^{\frac{1}{2}} = N\mathcal{L}^{A+\frac{3}{2}} e^{-\frac{1}{4}\sqrt{\mathcal{L}}} \ll N.$$

Recall that we write $n = n_1 n_2$ where n_1 consists of all prime factors of n in $(Y, Z]$. Define

$$D'_k(N, a) = \sum_{\substack{n \le N \\ \tau_k(n) \le \mathcal{L}^A \\ n_1 > 1}} \tau(n+a)\tau_k(n), \tag{2.4}$$

so it follows that

$$\begin{aligned} &D_k(N, a) - D'_k(N, a) \\ &= \Bigg(\sum_{\substack{n \le N \\ \tau_k(n) > \mathcal{L}^A}} + \sum_{\substack{n \le N \\ \tau_k(n) \le \mathcal{L}^A \\ p|n \Rightarrow p > Z}} + \sum_{\substack{n \le N \\ \tau_k(n) \le \mathcal{L}^A \\ p|n \Rightarrow p \le Y}} \Bigg) \tau(n+a)\tau_k(n) = O(N). \end{aligned}$$

□

Proof of Proposition 1.2. Given Proposition 1.1, it suffices to estimate the contribution of terms in $D'_k(N, a)$ with n_1 not square-free, which is

$$\sum_{\substack{n \le N \\ \tau_k(n) \le \mathcal{L}^A \\ \mu(n_1) = 0}} \tau(n+a)\tau_k(n) \le \mathcal{L}^A \sum_{Y < p \le Z} \sum_{\substack{a < m \le N + a \\ n \equiv a (p^2)}} \tau(m).$$

By Lemma 2.1, the right-hand side is

$$\ll \mathcal{L}^A \sum_{Y < p \le Z} \frac{N}{p^2} \mathcal{L} \ll N\mathcal{L}^{A+1} Y^{-1} = N\mathcal{L}^{A+1} e^{-\sqrt{\mathcal{L}}} \ll N.$$

□

2.2 Subdivision of $D_{k,\ell}(N, a)$

Recall that

$$D_{k,\ell}(N, a) = \sum_{Y < p \le Z} \sum_{m \le N/p} a_{m,\ell} \tau(pm + a),$$

where $a_{m,\ell}$ is a certain coefficient majorized by $\mathcal{L}^A$.

Let $H_1 > 0$ be some large constant and $\eta_1 = 1 + \mathcal{L}^{-H_1}$. Then the interval $(Y, Z]$ is subdivided into subintervals of the form $(D, \eta_1 D]$† with $D = \eta_1^j Z$ and

$$0 \le j \le \frac{\log(Z/Y)}{\log \eta_1} \ll \mathcal{L}^{H_1+1}.$$

†One of the interval may be incomplete, but our approach still applies

Thus, $D_{k,\ell}(N,a)$ is decomposed into

$$D_{k,\ell}(N,a;D)=\sum_{D<p\le\eta_1 D}\sum_{m\le N/p}a_{m,\ell}\tau(pm+a).$$

As p lies in a thin interval, we can get rid of the dependence of m on p by replacing N/p with N/D in the inner summation. If

$$S_{k,\ell}(N,a;D)=\sum_{D<p\le\eta_1 D}\sum_{m\le N/D}a_{m,\ell}\tau(pm+a),$$

then by Lemma 2.1, one has

$$\begin{aligned}|D_{k,\ell}(N,a;D)-S_{k,\ell}(N,a;D)|&\le\mathcal{L}^A\sum_{D<p\le\eta_1 D}\sum_{\frac{N}{\eta_1 D}<m\le\frac{N}{D}}\tau(pm+a)\\&\ll\mathcal{L}^A\sum_{D<p\le\eta_1 D}\frac{N}{D}(1-\eta_1^{-1})\mathcal{L}\ll N\mathcal{L}^{A+1-2H_1}.\end{aligned}$$

The contribution of this error in $D_{k,\ell}(N,a)$ is $\ll N\mathcal{L}^{A+2-H_1}$. From the statement of Proposition 1.2, we know that the total contribution in $D_k(N,a)$ is $O(N)$ if $H_1\ge A+\frac{3}{2}$.

Now, let $H_2\ge H_1$ be some large constant and $\eta_2=1+\mathcal{L}^{-H_2}$. We subdivide $[1,N/D]$ into $\ll\mathcal{L}^{H_2+1}$ intervals in a similar manner, thereby reducing $S_{k,\ell}(N,a;D)$ to

$$S_{k,\ell}(N,a;D,M)=\sum_{D<p\le\eta_1 D}\sum_{M<m\le\eta_2 M}a_{m,\ell}\tau(pm+a).$$

Trivially, we know

$$S_{k,\ell}(N,a;D,M)\ll DM\mathcal{L}^{1-H_2},$$

which is a very helpful when DM is small. For this reason, we assume

$$DM\ge N\mathcal{L}^{-\frac{1}{2}K_1}=P\mathcal{L}^{\frac{1}{2}K_1},\quad K_1\gg H_1+H_2 \tag{2.5}$$

in all subsequent discussions.

2.3 Preliminary treatment of the dispersion

Recall from §1.1 that

$$V(N,a;D,M)=\sum_{D<p_1,p_2\le\eta_1 D}(W(1,1;p_1,p_2)-2W(1,a;p_1,p_2)+W(a,a;p_1,p_2)), \tag{2.6}$$

where

$$W(a_1,a_2;p_1,p_2)=\sum_{M<m\le\eta_2 M}\tau(p_2m+a_1)\tau(p_1m+a_2). \tag{2.7}$$

Thus, to estimate $V(N,a;D,M)$, it suffices to give an asymptotic expansion of

$$\sum_{D<p_1,p_2\le\eta_1 D} W(a_1,a_2;p_1,p_2), \tag{2.8}$$

with $a_j=1$ or a prime $\sim P$. Because $P>Z$, we know $(a_1a_2,p_1p_2)=1$, allowing us to invoke Lemma 2.1 without pain, so trivially, one has

$$\begin{aligned} W(a_1,a_2;p_1,p_2) &\le \left(\sum_{M<m\le\eta_2 M}\tau(pm+a_1)^2\right)^{\frac12}\left(\sum_{M<m\le\eta_2 M}\tau(pm+a_2)^2\right)^{\frac12} \\ &\ll (M\mathcal{L}^{2^2-1-H_2})^{\frac12}(M\mathcal{L}^{2^2-1-H_2})^{\frac12}=M\mathcal{L}^{3-H_2}, \end{aligned}$$

so the total contribution of diagonal terms $p_1=p_2$ in (2.20) is

$$\ll DM\mathcal{L}^{3-H_1-H_2}.$$

It remains to treat the off-diagonal terms. By the symmetry in (2.7), we assume $p_1<p_2$ onwards.

$W(a_1,a_2;p_1,p_2)$ counts the number of $(x,y,z,t,m)\in\mathbb{N}^5$ satisfying

$$\text{(i) } M<m\le\eta_2 M,\quad \text{(ii) } p_2m+a_1=xy,\quad \text{(iii) } p_1m+a_2=zt. \tag{2.9}$$

It follows from (ii) and (iii) that

$$p_1xy-p_2zt=p_1a_1-p_2a_2,\quad m=\frac{xy-a_1}{p_2}.$$

Now, define $G=p_1a_1-p_2a_2$, so we can eliminate the m in (2.9) and reinterpret $W(a_1,a_2;p_1,p_2)$ as the number of $(x,y,z,t)\in\mathbb{N}^4$ such that

$$\text{(i) } p_1xy-p_2zt=G,\quad \text{(ii) } M<\frac{xy-a_1}{p_2}\le\eta_2 M. \tag{2.10}$$

Note that (ii) of (2.10) implies $xy,zt\le 2N$ and $xy=yx, zt=tz$, $W(a_1,a_2;p_1,p_2)$ may be well approximated by four times the following quantity:

$$\begin{aligned} &W'(a_1,a_2;p_1,p_2)=\text{number of } (x,y,z,t)\in\mathbb{N}^4 \text{ s.t.} \\ &\text{(i) } x\le\sqrt{2N},\quad x<y\le\frac{2N}{x};\quad \text{(ii) } z\le\sqrt{2N},\quad z<t\le\frac{2N}{z}; \\ &\text{(iii) } p_1xy-p_2zt=G,\quad \text{(iv) } M<\frac{xy-a_1}{p_2}\le\eta_2 M. \end{aligned} \tag{2.11}$$

Specifically, the error $W(a_1,a_2;p_1,p_2)-4W'(a_1,a_2;p_1,p_2)$ consists precisely of (x,y,z,t) in $W(a_1,a_2;p_1,p_2)$ such that $x=y$ or $z=t$, which is

$$\le \sum_{\substack{x\le\sqrt{2N}\\ p_1x^2\equiv G(p_2)}}\tau\left(\frac{p_1x^2-G}{p_2}\right)+\sum_{\substack{z\le\sqrt{2N}\\ p_2z^2\equiv -G(p_1)}}\tau\left(\frac{p_2z^2+G}{p_1}\right).$$

Combining this with the divisor bound, we have

$$W(a_1,a_2;p_1,p_2) = 4W'(a_1,a_2;p_1,p_2) + O(N^{\frac{1}{2}+\varepsilon}). \tag{2.12}$$

Let $W'(a_1,a_2;p_1,p_2,\delta)$ be the number of solutions in (2.11) with $(x,z)=\delta$. Then by (iii) of (2.11), we know $W'(a_1,a_2;p_1,p_2,\delta)=0$ whenever $\delta \nmid G$, so

$$\sum_{\substack{D<p_1,p_2\le\eta_1 D\\ p_1\ne p_2}} W'(a_1,a_2;p_1,p_2) = \sum_{\delta\ge 1}\sum_{\substack{D<p_1,p_2\le\eta_1 D\\ p_1\ne p_2\\ \delta\mid(p_1a_1-p_2a_2)}} W'(a_1,a_2;p_1,p_2,\delta) \tag{2.13}$$

We will show that the main contribution to the right-hand side comes from small values of δ.

When $p_1a_1-p_2a_2=G=G_1\delta$,

$$\begin{aligned}
&W'(a_1,a_2;p_1,p_2,\delta) = \text{number of } (x,y,z,t)\in\mathbb{N}^4 \text{ s.t.}\\
&\text{(i) } x,z\le\frac{\sqrt{2N}}{\delta},\quad (x,z)=1;\quad \text{(ii) } x\delta<y\le\frac{2N}{x\delta},\quad z\delta<t\le\frac{2N}{z\delta};\\
&\text{(iii) } p_1xy-p_2zt=G_1,\quad \text{(iv) } M<\frac{\delta xy-a_1}{p_2}\le\eta_2 M.
\end{aligned} \tag{2.14}$$

$W'(a_1,a_2;p_1,p_2,\delta)$ is bounded by the number of tuples (x,y,z,t) satisfying (iii) and (iv), so

$$\begin{aligned}
W'(a_1,a_2;p_1,p_2,\delta) &\le \sum_{\substack{\frac{p_2M+a_1}{\delta}<u\le\frac{p_2\eta_2M+a_1}{\delta}\\ p_1u\equiv G_1(p_2)}} \tau(u)\tau\left(\frac{p_1u-G_1}{p_2}\right)\\
&\le \left(\sideset{}{'}\sum_{u\equiv\overline{p_1}G_1(p_2)}\tau^2(u)\right)^{\frac{1}{2}}\left(\sideset{}{'}\sum_{p_1u\equiv G_1(p_2)}\tau^2\left(\frac{p_1u-G_1}{p_2}\right)\right)^{\frac{1}{2}},
\end{aligned}$$

in which the $\sum_u'$ sums over the interval $\frac{p_2M+a_1}{\delta}<u\le\frac{p_2\eta_2M+a_1}{\delta}$. Now, by Lemma 2.1, we know

$$\sideset{}{'}\sum_{u\equiv\overline{p_1}G_1(p_2)}\tau^2(u) \ll \frac{p_2M\mathcal{L}^{2^2-1-H_2}}{p_2\delta} = \frac{M}{\delta}\mathcal{L}^{3-H_2}.$$

As for the second summation, write $p_1u=G_1+p_2v$. Then

$$\frac{p_2M+a_1+G_1\delta}{p_2\delta}<v\le\frac{p_2M+a_1+G_1\delta}{p_2\delta}+\frac{M}{\delta}\mathcal{L}^{-H_2},$$

so it follows from Lemma 2.1 that this is $\ll\frac{M}{\delta}\mathcal{L}^{3-H_2}$ as well. Consequently,

$$W'(a_1,a_2;p_1,p_2,\delta)\ll\frac{M}{\delta}\mathcal{L}^{3-H_2}.$$

Let H_3 be some large constant. Then we see that the terms in (2.13) with $\delta > \mathcal{L}^{H_3}$ are bounded by

$$\begin{aligned}
&\ll \sum_{\delta > \mathcal{L}^{H_3}} \sum_{\substack{D < p_1, p_2 \le \eta_1 D \\ p_1 \ne p_2 \\ \delta | (p_1 a_1 - p_2 a_2)}} \frac{M\mathcal{L}^{3-H_2}}{\delta} = \sum_{\delta > \mathcal{L}^{H_3}} \frac{M\mathcal{L}^{3-H_2}}{\delta} \sum_{D < p_1 \le \eta_1 D} \sum_{\substack{D < p_2 \le \eta_1 D \\ p_2 \equiv \overline{a_2} p_1 a_1 (\delta)}} 1 \\
&\ll D^2 M \mathcal{L}^{3-2H_1-H_2} \sum_{\delta > \mathcal{L}^{H_3}} \frac{1}{\delta^2} \ll D^2 M \mathcal{L}^{3-2H_1-H_2-H_3} \le D^2 M \mathcal{L}^{-\frac{1}{2}H_3}
\end{aligned}$$

provided $H_3 \ge 2(2H_1 + H_2 - 3)$. Combining this bound with (2.12) and (2.13), we deduce that

$$\begin{aligned}
\sum_{D < p_1, p_2 \le \eta_1 D} W(a_1, a_2; p_1, p_2) = 4 &\sum_{\delta \le \mathcal{L}^{H_3}} \sum_{\substack{D < p_1, p_2 \le \eta_1 D \\ p_1 \ne p_2 \\ \delta | (p_1 a_1 - p_2 a_2)}} W'(a_1, a_2; p_1, p_2, \delta) \\
&+ O(D^2 M \mathcal{L}^{-\frac{1}{2}H_3} + D^2 N^{\frac{1}{2}+\varepsilon}).
\end{aligned} \tag{2.15}$$

Inspecting (2.14), we can eliminate the variable t by rewriting (iii) into

$$p_1 xy \equiv G_1 \pmod{p_2 z}, \quad t = \frac{p_1 xy - G_1}{p_2 z}.$$

Combining this with (ii), we have

$$z\delta < t = \frac{p_1 xy - G_1}{p_2 z} \Rightarrow z^2 < \frac{p_1 xy - G_1}{p_2 \delta},$$

and because

$$\frac{p_1 xy - G_1}{p_2 z} \le \frac{xy}{z} \le \frac{2N}{\delta z},$$

the upper bound $t \le \frac{2N}{\delta z}$ in (ii) of (2.14) is automatically satisfied. Consequently, $W'(a_1, a_2; p_1, p_2, \delta)$ counts the number of $(x, y, z) \in \mathbb{N}^3$ such that

$$\begin{aligned}
&\text{(i) } x, z \le \frac{\sqrt{2N}}{\delta}, \quad (x, z) = 1; \quad \text{(ii) } x\delta < y \le \frac{2N}{x\delta}, \\
&\text{(iii) } p_1 xy \equiv G_1 \pmod{p_2 z}, \quad \text{(iv) } z^2 < \frac{p_1 xy - G_1}{p_2 \delta}, \\
&\text{(v) } \frac{p_2 M + a_1}{x\delta} < y \le \frac{p_2 \eta_2 M + a_1}{x\delta}.
\end{aligned} \tag{2.16}$$

Note that (v) of (2.16) is equivalent to

$$M < \frac{\delta xy - a_1}{p_2} \le \eta_2 M.$$

Recall that $G_1 \delta = p_1 a_1 - p_2 a_2$, so

$$\frac{p_1 xy - G_1}{p_2 \delta} = \frac{p_1 x\delta y - p_1 a_1 + p_2 a_2}{p_2 \delta^2} = \frac{p_1}{\delta^2} \frac{\delta xy - a_1}{p_2} + \frac{a_2}{\delta^2}$$

is approximately DM/δ^2. Therefore, it is reasonable to replace (iv) with $z \le \sqrt{DM}/\delta$. Indeed, the error of such replacement is bounded by the number of $(x, y, z) \in \mathbb{N}^3$ such that

$$\begin{aligned}&\text{(i) } \frac{\sqrt{DM}}{\delta} < z \le \frac{\sqrt{\eta_1\eta_2 DM + a_2}}{\delta}, \quad \text{(ii) } x \le \frac{\sqrt{2N}}{\delta}, \quad (x, z) = 1;\\ &\text{(iii) } p_1 xy \equiv G_1 \pmod{p_2 z}, \quad \text{(iv) } \frac{p_2 M + a_1}{\delta} < xy \le \frac{p_2\eta_2 M + a_1}{\delta}.\end{aligned} \tag{2.17}$$

If p_1 is not invertible modulo $p_2 z$, then we must have $p_1 | (p_2 z)$, so p_1 divides $G_1\delta = p_1 a_1 - p_2 a_2$, thereby dividing a_2. However, this is impossible as a_2 has no prime divisors in $(Y, Z]$. Therefore, the number of (x, y, z) satisfying (2.17) is

$$\begin{aligned}&\le \sum_{\frac{\sqrt{DM}}{\delta} < z \le \frac{\sqrt{\eta_1\eta_2 DM + a_2}}{\delta}} \sum_{\substack{\frac{p_2 M + a_1}{\delta} < u \le \frac{p_2\eta_2 M + a_1}{\delta} \\ m \equiv \overline{p_1} G_1 (p_2 z)}} \tau(m)\\ &\ll \sum_{\frac{\sqrt{DM}}{\delta} < z \le \frac{\sqrt{\eta_1\eta_2 DM + a_2}}{\delta}} \frac{p_2 M \mathcal{L}^{1-H_2}}{\delta p_2 z}\\ &= \frac{M}{\delta}\mathcal{L}^{1-H_2} \sum_{\frac{\sqrt{DM}}{\delta} < z \le \frac{\sqrt{\eta_1\eta_2 DM + a_2}}{\delta}} \frac{1}{z} \ll \frac{M}{\delta}\mathcal{L}^{1-H_2} \log\left(\eta_1\eta_2 + \frac{a_2}{DM}\right)\end{aligned}$$

Since it was specified above that $H_2 \ge H_1$, we have $\eta_1\eta_2 = 1 + O(\mathcal{L}^{-H_1})$. By (2.5), we know $a_2/DM \ll \mathcal{L}^{-K} \le \mathcal{L}^{-H_1}$, so this bound simplifies into

$$\ll \frac{M}{\delta}\mathcal{L}^{1-H_1-H_2}.$$

Summing over $\delta \le \mathcal{L}^{H_3}$, this contributes $\ll M\mathcal{L}^{1-H_1-H_2}\log\mathcal{L}$ to (2.15). As $D \ge Y = \exp(\sqrt{\mathcal{L}})$, this error is completely absorbed in the remainder of (2.15).

Note that (ii) of (2.16) is equivalent to

$$\frac{\delta xy - a_1}{p_2} > \frac{\delta x^2 - a_1}{p_2},$$

which is again equivalent to

$$x^2 < \frac{1}{\delta^2}\left(p_2 \cdot \frac{\delta xy - a_1}{p_2} + a_1\right).$$

Therefore, similar to how we treat z, replacing (ii) of (2.16) with $x \le \sqrt{DM}/\delta$ also results in an error that can be absorbed in the remainder of (2.15).

Collecting these observations, we deduce the following

Proposition 2.1. *Let $W''(a_1, a_2; p_1, p_2, \delta)$ be the number of $(x, y, z) \in \mathbb{N}^3$ such that*

$$\begin{aligned} &\textit{(i) } x, z \le \frac{\sqrt{DM}}{\delta}, \quad (x, z) = 1; \quad \textit{(ii) } p_1 xy \equiv G_1 \pmod{p_2 z}, \\ &\textit{(iii) } \frac{p_2 M + a_1}{\delta x} < y \le \frac{p_2 \eta_2 M + a_1}{\delta x}. \end{aligned} \tag{2.18}$$

Then one has

$$\sum_{D < p_1, p_2 \le \eta_1 D} W(a_1, a_2; p_1, p_2) = 4 \sum_{\delta \le \mathcal{L}^{H_3}} \sum_{\substack{D < p_1, p_2 \le \eta_1 D \\ p_1 \ne p_2 \\ \delta | (p_1 a_1 - p_2 a_2)}} W''(a_1, a_2; p_1, p_2, \delta) + O(D^2 M \mathcal{L}^{-\frac{1}{2} H_3} + D^2 N^{\frac{1}{2} + \varepsilon}). \tag{2.19}$$

2.4 Asymptotic formula for $W''(a_1, a_2; p_1, p_2, \delta)$

From Proposition 2.1, we can rewrite $W''(a_1, a_2; p_1, p_2, \delta)$ as

$$W''(a_1, a_2; p_1, p_2, \delta) = \sum_{\substack{x \le \sqrt{DM}/\delta \\ (x, p_2 z) = 1}} \sum_{z \le \sqrt{DM}/\delta} \sum_{\substack{\frac{p_2 M + a_1}{\delta x} < y \le \frac{p_2 \eta_2 M + a_1}{\delta x} \\ y \equiv \overline{p_1 x} G_1 (p_2 z)}} 1, \tag{2.20}$$

where the requirement $(x, p_2 z) = 1$ follows from the fact that the right-hand side of the congruence (ii) in Proposition 2.1 has no solutions in y when p_2 divides x.

Trivially, the inner sum evaluates to

$$\frac{M \mathcal{L}^{-H_2}}{\delta x z} + O(1),$$

which is not suitable when xz is large. As a result, we perform harmonic analysis on the error term.

Lemma 2.5. *Define $\psi(\theta) = \theta - \lfloor \theta \rfloor - \frac{1}{2}$. Then for real numbers Y, Y' and integers h, q,*

$$\sum_{\substack{Y < y \le Y' \\ y \equiv h(q)}} 1 = \frac{Y' - Y}{q} + \psi\left(\frac{Y - h}{q}\right) - \psi\left(\frac{Y' - h}{q}\right).$$

Proof. Write $y = qt + h$. Then the left-hand side becomes

$$\sum_{\frac{Y-h}{q} < t \leq \frac{Y'-h}{q}} 1 = \left\lfloor \frac{Y'-h}{q} \right\rfloor - \left\lfloor \frac{Y-h}{q} \right\rfloor$$
$$= \frac{Y'-Y}{q} + \left\lfloor \frac{Y'-h}{q} \right\rfloor - \frac{Y'-h}{q} - \left(\left\lfloor \frac{Y-h}{q} \right\rfloor - \frac{Y-h}{q} \right)$$
$$= \frac{Y'-Y}{q} + \psi\left(\frac{Y-h}{q}\right) - \psi\left(\frac{Y'-h}{q}\right).$$

□

Lemma 2.6 (Erdös-Turán). *Let $u_1, u_2, \dots, u_N \in \mathbb{R}$. Then for all $\Delta > 0$,*

$$\left| \sum_{n=1}^{N} \psi(u_n) \right| \ll \Delta n + \sum_{h \leq \Delta^{-1}} \frac{1}{h} \left| \sum_{n=1}^{N} e(hu_n) \right|.$$

Proof. Set $F(\alpha) = u - \lfloor u \rfloor$ and plug [19, Corollary 1.1] into [19, (13)]. □

We also need some results on expoential sums:

Lemma 2.7. *Define the Ramanujan sum as*

$$c_q(b) = \sum_{\substack{0 \leq x < q \\ (x,q)=1}} e\left(\frac{bx}{q}\right).$$

Then one has $|c_q(b)| \leq (b, q)$.

Proof. By Möbius inversion, we see that when $\Delta = (b, q), b = b'\Delta, q = q'\Delta$,

$$c_q(b) = \sum_{d|q} \mu(d) \sum_{\substack{0 \leq x < q \\ d|x}} e\left(\frac{bx}{q}\right) = \sum_{d|q} \mu(d) \sum_{0 \leq y < q/d} e\left(\frac{by}{q/d}\right)$$
$$= \sum_{d|q, q/d|b} \mu(d) \frac{q}{d} = \sum_{t|\Delta} t\mu\left(\frac{q}{t}\right) = \Delta \sum_{u|\Delta} \frac{\mu(q'u)}{u}$$
$$= \Delta\mu(q') \sum_{\substack{u|\Delta \\ (u,q')=1}} \frac{\mu(t)}{t} = \Delta\mu(q') \prod_{p|\Delta, p\nmid q'} \left(1 - \frac{1}{p}\right).$$

Bounding the last term trivially, we obtain the desired result. □

Lemma 2.8 (Weil bound [13, Corollary 11.12]). *Define the Kloosterman sum as*

$$S(a, b; q) = \sum_{\substack{0 \leq x < q \\ (x,q)=1}} e\left(\frac{ax + b\overline{x}}{q}\right).$$

Then one has $|S(a, b; q)| \leq q^{\frac{1}{2}} \tau(q) (a, b, q)^{\frac{1}{2}}$.

Lemma 2.9. *Let $q \in \mathbb{N}$ and $(b,q)=1$. Then for integers $V>U$,*

$$\sum_{\substack{U<x\le V\\(x,q)=1}} e\left(\frac{b\overline{x}}{q}\right) \ll \frac{V-U}{q}(b,q) + q^{\frac12}\tau(q)(b,q)^{\frac12}\log(2q).$$

Proof. By periodicity, the left-hand side is

$$\begin{aligned}
&= \sum_{\substack{0\le r<q\\(r,q)=1}} e\left(\frac{b\overline{r}}{q}\right) \sum_{\substack{U<x\le V\\ m\equiv r(q)}} 1 = \frac1q \sum_{\substack{0\le r<q\\(r,q)=1}} e\left(\frac{b\overline{r}}{q}\right) \sum_{U<x\le V}\sum_{0\le s<q} e\left(\frac{s(r-x)}{q}\right)\\
&= \frac1q \sum_{U<x\le V}\sum_{0\le s<q} e\left(-\frac{sx}{q}\right) \sum_{\substack{0\le r<q\\(r,q)=1}} e\left(\frac{sr+b\overline{r}}{q}\right)\\
&= \frac1q \sum_{U<x\le V}\sum_{0\le s<q} e\left(-\frac{sx}{q}\right) S(s,b;q).
\end{aligned}$$

Observe that $S(0,b;q)=c_q(b)$, the last term evaluates to

$$\begin{aligned}
&= \frac1q \sum_{U<x\le V} c_q(b) + \frac1q \sum_{1\le s<q}\sum_{U<x\le V} e\left(-\frac{sx}{q}\right) S(s,b;q)\\
&\ll \frac{V-U}{q}|c_q(b)| + q^{-\frac12}\tau(q)\sum_{1\le s<q}(b,s,q)^{\frac12}\left|\sum_{U<x\le V} e\left(-\frac{sx}{q}\right)\right|. \qquad (2.21)
\end{aligned}$$

By Lemma 2.7, the first term is $\ll \frac{V-U}{q}(b,q)$.

As for the second term, it follows from the properties of geometric progression that

$$\begin{aligned}
\left|\sum_{U<x\le V} e\left(-\frac{sx}{q}\right)\right| &= \left|e\left(-\frac{(U+1)x}{q}\right)\sum_{0\le y<V-U} e\left(-\frac{sy}{q}\right)\right| = \left|\frac{1-e(-(U-V)s/q)}{1-e(-s/q)|}\right|\\
&= \left|\frac{\sin(\pi(U-V)s/q)}{\sin(\pi s/q)}\right| \le \frac{2}{\|s/q\|},
\end{aligned}$$

where $\|\cdot\|$ denotes the distance to the closest integer. When $s=1,2,\dots,q$, $\|s/q\|$ takes each of $1/q, 2/q, \dots, \lfloor q/2\rfloor/q$ exactly twice. Therefore,

$$\begin{aligned}
\sum_{1\le s<q}(b,s,q)^{\frac12}\left|\sum_{U<x\le V} e\left(-\frac{sx}{q}\right)\right| &\le (b,q)^{\frac12}\sum_{1\le s<q}\frac{2}{\|s/q\|}\\
&= 4(b,q)^{\frac12}\sum_{1\le s\le q/2}\frac{q}{s} \ll q(b,q)^{\frac12}\log(2q).
\end{aligned}$$

Therefore, the second term in (2.21) is $\ll q^{\frac12}\tau(q)(b,q)^{\frac12}\log(2q)$. □

Now, let's work out the asymptotic formula for $W''(a_1, a_2; p_1, p_2, \delta)$. Applying Lemma 2.5 to the inner sum of (2.20), we see that the main term is

$$\sum_{\substack{x \le \sqrt{DM}/\delta \\ (x, p_2 z)=1}} \sum_{z \le \sqrt{DM}/\delta} \frac{p_2 M \mathcal{L}^{-H_2}}{p_2 \delta x z} = \frac{M \mathcal{L}^{-H_2}}{\delta} \sum_{\substack{x \le \sqrt{DM}/\delta \\ (x, p_2 z)=1}} \sum_{z \le \sqrt{DM}/\delta} \frac{1}{xz}, \tag{2.22}$$

and the error consists of two terms of the form

$$\sum_{\substack{x \le \sqrt{DM}/\delta \\ (x, p_2 z)=1}} \sum_{z \le \sqrt{DM}/\delta} \psi\left(\frac{L - \overline{p_1 x} G_1}{p_2 z}\right), \tag{2.23}$$

with $\frac{p_2 M + a_1}{\delta x}$ and $\frac{p_2 \eta_2 M + a_1}{\delta x}$ in place of L respectively.

By Lemma 2.6, the error (2.20) becomes

$$\ll \frac{DM}{\delta^2} \Delta + \sum_{h \le \Delta^{-1}} \frac{|E_h|}{h},$$

where

$$E_h = \sum_{\substack{x \le \sqrt{DM}/\delta \\ (x, p_2 z)=1}} \sum_{z \le \sqrt{DM}/\delta} e\left(h \cdot \frac{L - \overline{p_1 x} G_1}{p_2 z}\right).$$

By Lemma 2.9 (x is automatically coprime to p_2), one has

$$\begin{aligned}
|E_h| &= \Bigg| \sum_{z \le \sqrt{DM}/\delta} e\left(\frac{hL}{p_2 z}\right) \sum_{\substack{x \le \sqrt{DM}/\delta \\ (x, p_2 z)=1}} e\left(-\frac{h \overline{p_1 x} G_1}{p_2 z}\right) \Bigg| \\
&\le \sum_{z \le \sqrt{DM}/\delta} \Bigg| \sum_{\substack{x \le \sqrt{DM}/\delta \\ (x, p_2 z)=1}} e\left(-\frac{h \overline{p_1 x} G_1}{p_2 z}\right) \Bigg| \\
&\ll \sum_{z \le \sqrt{DM}/\delta} \left((G_1 h, p_2 z) \frac{\sqrt{DM}}{\delta p_2 z} + (p_2 z)^{\frac{1}{2}} (G_1 h, p_2 z)^{\frac{1}{2}} \tau(p_2 z) \mathcal{L} \right) \\
&= \Sigma_1 + \Sigma_2,
\end{aligned}$$

in which

$$\begin{aligned}
\Sigma_1 &\ll \frac{1}{\delta} \sqrt{\frac{M}{D}} \sum_{z \le \sqrt{DM}/\delta} \frac{(G_1 h, p_2 z)}{z} \le \frac{(G_1 h, p_2)}{\delta} \sqrt{\frac{M}{D}} \sum_{z \le \sqrt{DM}/\delta} \frac{(G_1 h, z)}{z} \\
&\le \frac{(G_1 h, p_2)}{\delta} \sqrt{\frac{M}{D}} \sum_{t | G_1 h} t \sum_{\substack{z \le \sqrt{DM}/\delta \\ t | z}} \frac{1}{z} = \frac{(G_1 h, p_2)}{\delta} \sqrt{\frac{M}{D}} \sum_{t | G_1 h} t \sum_{u \le \sqrt{DM}/(\delta t)} \frac{1}{tu} \\
&\ll \frac{(G_1 h, p_2)}{\delta} \sqrt{\frac{M}{D}} \tau(G_1 h) \mathcal{L} \ll \frac{(G_1 h, p_2)}{\delta} \sqrt{\frac{M}{D}} N^{\varepsilon}
\end{aligned}$$

and

$$\begin{aligned}\Sigma_2 &\ll D^{\frac{1}{2}}(G_1h,p_2)^{\frac{1}{2}}N^{\frac{1}{2}\varepsilon}\sum_{z\le\sqrt{DM}/\delta} z^{\frac{1}{2}}(G_1h,z)^{\frac{1}{2}}\\ &\ll D^{\frac{1}{2}}(G_1h,p_2)^{\frac{1}{2}}N^{\frac{1}{2}\varepsilon}\sum_{t|G_1h} t^{\frac{1}{2}}\sum_{u\le\sqrt{DM}/(\delta t)}(ut)^{\frac{1}{2}}\\ &\ll D^{\frac{1}{2}}(G_1h,p_2)^{\frac{1}{2}}N^{\frac{1}{2}\varepsilon}\sum_{t|G_1h} t^{1-\frac{3}{2}}\frac{(DM)^{\frac{3}{4}}}{\delta^{\frac{3}{2}}}\\ &\ll D^{\frac{5}{4}}M^{\frac{3}{4}}(G_1h,p_2)^{\frac{1}{2}}\delta^{-\frac{3}{2}}N^{\varepsilon}.\end{aligned}$$

Therefore, we have

$$E_h \ll D^{-\frac{1}{2}}M^{\frac{1}{2}}\delta^{-1}(G_1h,p_2)N^{\varepsilon} + D^{\frac{5}{4}}M^{\frac{3}{4}}(G_1h,p_2)\delta^{-\frac{3}{2}}N^{\varepsilon}.$$

Since $p_2 \nmid G_1$, $(G_1h,p_2)=(h,p_2)$. Notice that

$$\begin{aligned}\sum_{h\le\Delta^{-1}}\frac{(G_1h,p_2)}{h} &= \sum_{\substack{h\le\Delta^{-1}\\ p_2|h}}\frac{p_2}{h} + \sum_{\substack{h\le\Delta^{-1}\\ p_2\nmid h}}\frac{1}{h}\\ &= \sum_{k\le\Delta^{-1}/p_2}\frac{p_2}{p_2k} + \sum_{\substack{h\le\Delta^{-1}\\ p_2\nmid h}}\frac{1}{h} \ll \log\Delta^{-1},\end{aligned}$$

so when $\Delta = N^{-2}$, the error (2.23) is

$$\ll D^{-\frac{1}{2}}M^{\frac{1}{2}}N^{2\varepsilon} + D^{\frac{5}{4}}M^{\frac{3}{4}}N^{2\varepsilon}.$$

Combining this with (2.22), we deduce that (renaming ε)

Proposition 2.2. *Let $W''(a_1,a_2;p_1,p_2,\delta)$ be defined as in Proposition 2.1. Then we have*

$$\begin{aligned}W''(a_1,a_2;p_1,p_2,\delta) = \frac{M\mathcal{L}^{-H_2}}{\delta}\sum_{\substack{x\le\sqrt{DM}/\delta\\ (x,p_2z)=1}}\sum_{z\le\sqrt{DM}/\delta}\frac{1}{xz}\\ + O(D^{-\frac{1}{2}}M^{\frac{1}{2}}N^{\varepsilon} + D^{\frac{5}{4}}M^{\frac{3}{4}}N^{\varepsilon}).\end{aligned}$$

2.5 Upper bound for the dispersion

Proof of Proposition 1.3. The contribution of the main term from Proposition 2.2 to (2.19) is

$$\begin{aligned}\Sigma_0 &= \sum_{\delta\le\mathcal{L}^{H_3}} \frac{4M\mathcal{L}^{-H_2}}{\delta} \sum_{\substack{D<p_1,p_2\le\eta_1 D\\ p_1\ne p_2\\ \delta|(p_1a_1-p_2a_2)}} \sum_{\substack{x\le\sqrt{DM}/\delta\\ (x,p_2z)=1}} \sum_{z\le\sqrt{DM}/\delta} \frac{1}{xz}\\ &= \sum_{\delta\le\mathcal{L}^{H_3}} \frac{4M\mathcal{L}^{-H_2}}{\delta} \sum_{z\le\sqrt{DM}/\delta} \frac{1}{z} \sum_{D<p_1\le\eta_1 D} \sum_{\substack{D<p_2\le\eta_1 D\\ p_2\equiv\overline{a_2}p_1a_1(\delta)}} \sum_{\substack{x\le\sqrt{DM}/\delta\\ (x,p_2z)=1}} \frac{1}{x}.\end{aligned}$$

The condition $(x,p_2z)=1$ in the inner sum prevents us from interchanging the summation freely. It turns out that replacing $(x,p_2z)=1$ with $(x,z)=1$ creates a negligible error. Define

$$\begin{aligned}\Sigma_0^* &= \sum_{\delta\le\mathcal{L}^{H_3}} \frac{4M\mathcal{L}^{-H_2}}{\delta} \sum_{z\le\sqrt{DM}/\delta} \frac{1}{z} \sum_{D<p_1\le\eta_1 D} \sum_{\substack{D<p_2\le\eta_1 D\\ p_2\equiv\overline{a_2}p_1a_1(\delta)}} \sum_{\substack{x\le\sqrt{DM}/\delta\\ (x,z)=1}} \frac{1}{x}\\ &= \sum_{\delta\le\mathcal{L}^{H_3}} \frac{4M\mathcal{L}^{-H_2}}{\delta} \sum_{\substack{x\le\sqrt{DM}/\delta\\ (x,z)=1}} \sum_{z\le\sqrt{DM}/\delta} \frac{1}{xz} \sum_{D<p_1\le\eta_1 D} \sum_{\substack{D<p_2\le\eta_1 D\\ p_2\equiv\overline{a_2}p_1a_1(\delta)}} 1.\end{aligned}$$

Then one has

$$\begin{aligned}|\Sigma_0-\Sigma_0^*| &\ll \sum_{\delta\le\mathcal{L}^{H_3}} \frac{M\mathcal{L}^{-H_2}}{\delta} \sum_{z\le\sqrt{DM}/\delta} \frac{1}{z} \sum_{D<p_1\le\eta_1 D} \sum_{\substack{D<p_2\le\eta_1 D\\ p_2\equiv\overline{a_2}p_1a_1(\delta)}} \frac{\mathcal{L}}{D}\\ &\ll \sum_{\delta\le\mathcal{L}^{H_3}} \frac{M\mathcal{L}^{-H_2}}{\delta} \sum_{z\le\sqrt{DM}/\delta} \frac{1}{z}\frac{\mathcal{L}^{1-2H_1}D^2}{D\delta} \ll DM\mathcal{L}^{2-2H_1-H_2} \sum_{\delta\le\mathcal{L}^{H_3}} \frac{1}{\delta^2}\\ &\ll DM\mathcal{L}^{2-2H_1-H_2},\end{aligned}$$

which can be absorbed into the error term of (2.19).

Because $\delta\le\mathcal{L}^{H_3}$, it follows from the Siegel-Walfisz theorem that

$$\sum_{\substack{D<p_2\le\eta_1 D\\ p_2\equiv\overline{a_2}p_1a_1(\delta)}} 1 = \frac{1}{\varphi(\delta)}\int_D^{\eta_1 D}\frac{\mathrm{d}u}{\log u} + O(D\mathcal{L}^{-100H_3}).$$

Hence, by the prime number theorem, one has

$$\sum_{D<p_1\le\eta_1 D} \sum_{\substack{D<p_2\le\eta_1 D\\ p_2\equiv\overline{a_2}p_1a_1(\delta)}} 1 = \frac{1}{\varphi(\delta)}\left(\int_D^{\eta_1 D}\frac{\mathrm{d}u}{\log u}\right)^2 + O(D^2\mathcal{L}^{-50H_3}).$$

This indicates that Σ_0^* becomes

$$\Sigma_0^* = 4\left(\int_D^{\eta_1 D} \frac{\mathrm{d}u}{\log u}\right)^2 M\mathcal{L}^{-H_2} \sum_{\delta \le \mathcal{L}^{H_3}} \frac{1}{\delta\varphi(\delta)} \sum_{\substack{x \le \sqrt{DM}/\delta \\ (x,z)=1}} \sum_{z \le \sqrt{DM}/\delta} \frac{1}{xz}$$
$$+ O(D^2 M\mathcal{L}^{-20H_3}).$$

The contribution of error terms from Proposition 2.2 to (2.19) is

$$\ll D^{\frac{5}{2}} M^{\frac{1}{2}} N^{\varepsilon} + D^{\frac{13}{4}} M^{\frac{3}{4}} N^{\varepsilon} \ll D^4 M^{\frac{3}{4}} N^{\varepsilon}.$$

Combining this with the remainder of (2.19) completes the proof. □

2.6 Deduction of the averaging phenomenon

Proof of Proposition 1.4. Recall from §1.1 that the bound for the dispersion implies that when a is a prime $\sim P$,

$$|S_{k,\ell}(N,1;D,M) - S_{k,\ell}(N,a;D,M)|$$
$$\ll DM\mathcal{L}^{-\frac{1}{3}H_3} + D^2 M^{\frac{7}{8}} N^{\varepsilon} + DM^{\frac{1}{2}} N^{\frac{1}{4}+\varepsilon}.$$

By §2.2, we know $D_{k,\ell}(N,1) - D_{k,\ell}(N,a)$ is dominated by

$$\ll \mathcal{L}^{H_1+H_2+2} \max_{D \le Z, M \le N/D} \left(DM\mathcal{L}^{-\frac{1}{3}H_3} + D^2 M^{\frac{7}{8}} N^{\varepsilon} + DM^{\frac{1}{2}} N^{\frac{1}{4}+\varepsilon}\right)$$
$$\ll N\mathcal{L}^{H_1+H_2+2-\frac{1}{3}H_3} + Z^2 N^{\frac{7}{8}+2\varepsilon} + ZN^{\frac{3}{4}+2\varepsilon}$$
$$\le N\mathcal{L}^{-\frac{1}{4}H_3} + N^{\frac{7}{8}+2u_0+2\varepsilon} + N^{\frac{3}{4}+u_0+2\varepsilon}.$$

Plugging this bound into Proposition 1.2 gives

$$\ll N\mathcal{L}^{-\frac{1}{5}H_3} + N^{\frac{7}{8}+2u_0+3\varepsilon} + N^{\frac{3}{4}+u_0+3\varepsilon}.$$

When $u_0 = \frac{1}{17}$ and $\varepsilon > 0$ is small, exponents of the second and the third terms are strictly < 1, so the entire quantity is $\ll N$, concluding the proof. □

3 Solution to the ternary problem

In this section, we give an asymptotic formula for the number $X_k(N)$ of solutions to the ternary problem stated in §1.2. Specifically, Proposition 1.6 is deduced in §3.1 and Proposition 1.8 is proved in §3.4.

3.1 Major arc estimates

Recall from (1.5) that

$$T_k(\alpha; x) = \sum_{n \le x} \tau_k(n) e(n\alpha).$$

By Perron's formula, we know when $2 \le T \le x$ and $c_0 = 1 + (\log x)^{-1}$, one has

$$T_k(\alpha; x) = \frac{1}{2\pi i} \int_{c_0 - iT}^{c_0 + iT} \frac{x^s}{s} D_k(s; \alpha) \mathrm{d}s + O\left(\frac{x^{1+\varepsilon}}{T}\right), \tag{3.1}$$

where

$$D_k(s; \alpha) = \sum_{n \ge 1} \frac{\tau_k(n) e(n\alpha)}{n^s}. \tag{3.2}$$

To estimate $T_k(\alpha; N)$ in $\mathfrak{M}(a/q)$ with q squarefree, it suffices to first study the case $\alpha = \frac{a}{q}$ and then apply partial summation. In order to achieve this, we need to explore the analytic properties of $D_k(s; \alpha)$ when $\alpha = \frac{a}{q}$.

Lemma 3.1. *Let $\zeta(s, \alpha)$ be the Hurwitz zeta function. Then for $(a, q) = 1$,*

$$D_k\left(s; \frac{a}{q}\right) = \sum_{1 \le b_1, b_2, \ldots, b_k \le q} e\left(\frac{a b_1 b_2 \cdots b_k}{q}\right) q^{-ks} \prod_{j=1}^{k} \zeta\left(s, \frac{b_j}{q}\right).$$

Proof. This follows directly from the definition of $\tau_k(n)$ and the observation that

$$\sum_{\substack{n \ge 1 \\ n \equiv b(q)}} \frac{1}{n^s} = q^{-s} \zeta\left(s, \frac{b}{q}\right).$$

□

Lemma 3.2. *Define*

$$M_k(s; q) = \sum_{\substack{1 \le a \le q \\ (a,q)=1}} D_k\left(s; \frac{a}{q}\right). \tag{3.3}$$

Then for $q > 1, (a, q) = 1$,

$$D_k\left(s; \frac{a}{q}\right) - \frac{1}{\varphi(q)} M_k(s; q)$$

is an entire function.

Proof. When $k = 1$, it follows from Lemma 3.1 that

$$D_1\left(s; \frac{a}{q}\right) = \sum_{1 \le b \le q} \frac{e(ab/q)}{q^s} \zeta\left(s, \frac{b}{q}\right).$$

Recall that

$$\zeta(s, \alpha) - \frac{1}{s-1}$$

is entire and when $q \nmid a$,

$$\sum_{1 \le b \le q} e\left(\frac{ab}{q}\right) = 0,$$

so

$$D_1\left(s; \frac{a}{q}\right) = \sum_{1 \le b \le q} \frac{e(ab/q)}{q^s} \left[\zeta\left(s, \frac{b}{q}\right) - \frac{1}{s-1}\right]$$

is entire whenever $q \nmid a$, thereby proving the $k = 1$ case.

Now, assume $k \ge 2$ and the conclusion is true for $k - 1$. We have

$$\begin{aligned} D_k\left(s; \frac{a}{q}\right) &= \sum_{m,d \ge 1} \frac{\tau_{k-1}(d) e(amd/q)}{(md)^s} = \sum_{1 \le \ell \le q} D_{k-1}\left(s; \frac{a\ell}{q}\right) \sum_{\substack{m \ge 1 \\ m \equiv \ell (q)}} \frac{1}{m^s} \\ &= \frac{1}{q} \sum_{1 \le \ell \le q} D_{k-1}\left(s; \frac{a\ell}{q}\right) \sum_{1 \le b \le q} e\left(-\frac{b\ell}{q}\right) D_1\left(s; \frac{b}{q}\right) \\ &= \frac{1}{q} \sum_{1 \le b \le q} D_1\left(s; \frac{b}{q}\right) \sum_{1 \le \ell \le q} e\left(-\frac{b\ell}{q}\right) D_{k-1}\left(s; \frac{a\ell}{q}\right). \end{aligned}$$

The contribution of terms with $b = q$ is

$$\begin{aligned} \frac{1}{q} \sum_{1 \le \ell \le q} e\left(-\frac{b\ell}{q}\right) D_{k-1}\left(s; \frac{a\ell}{q}\right) &= \frac{1}{q} \sum_{1 \le \ell \le q} D_{k-1}\left(s; \frac{a\ell}{q}\right) \\ &= \frac{1}{q} \sum_{n \ge 1} \frac{\tau_{k-1}(n)}{n^s} \sum_{1 \le \ell \le q} e\left(\frac{a\ell n}{q}\right) \\ &= \sum_{d \ge 1} \frac{\tau_{k-1}(qd)}{(qd)^s}, \end{aligned}$$

which does not depend on the specific choice of a. Therefore,

$$\begin{aligned} D_k\left(s; \frac{a}{q}\right) &= \sum_{d \ge 1} \frac{\tau_{k-1}(qd)}{(qd)^s} \\ &\quad + \frac{1}{q} \sum_{1 \le b < q} D_1\left(s; \frac{b}{q}\right) \sum_{h \mid q} \sum_{\substack{1 \le \ell \le h \\ (\ell, h) = 1}} e\left(-\frac{b\ell}{h}\right) D_{k-1}\left(s; \frac{a\ell}{h}\right). \end{aligned}$$

Now, observe that for $h|q, (a_0, h) = 1$,

$$\#\{1 \le a \le q : (a,q) = 1, a \equiv a_0 \pmod h\} = \frac{\varphi(q)}{\varphi(h)},$$

so

$$\frac{1}{\varphi(q)} \sum_{\substack{1\le a\le q\\(a,q)=1}} D_{k-1}\left(s; \frac{a\ell}{h}\right) = \frac{1}{\varphi(h)} \sum_{\substack{1\le a_0\le h\\(a,h)=1}} D_{k-1}\left(s; \frac{a_0\ell}{h}\right)$$
$$= \frac{1}{\varphi(h)} M_{k-1}(s;h).$$

Therefore, we have

$$D_k\left(s; \frac{a}{q}\right) - \frac{1}{\varphi(q)} M_k(s;q) = \frac{1}{q} \sum_{1\le b<q} D_1\left(s; \frac{b}{q}\right) \sum_{h|q} \sum_{\substack{1\le \ell\le h\\(\ell,h)=1}} e\left(-\frac{b\ell}{h}\right)$$
$$\times \left[D_{k-1}\left(s; \frac{a\ell}{h}\right) - \frac{1}{\varphi(h)} M_{k-1}(s;h)\right].$$

As $(a\ell, h) = 1$, the conclusion follows from the inductive hypothesis. □

Lemma 3.3. *Let $\eta > 0$ be fixed. Then for $\eta \le \sigma \le 1$, $|t| \ge 2$, and $\alpha \in (0,1]$, one has*

$$\zeta(s;\alpha) \ll \alpha^{-\sigma} + |t|^{1-\sigma} \log |t|.$$

Proof. When $\psi(\theta) = \theta - \lfloor\theta\rfloor - \frac{1}{2}$ and $\sigma > 1$, it follows from the Euler–Maclaurin formula that

$$\begin{aligned}\zeta(s,\alpha) &= \sum_{n=0}^{N} \frac{1}{(n+\alpha)^s} + \sum_{n>N} \frac{1}{(n+\alpha)^s}\\
&= \sum_{n=0}^{N} \frac{1}{(n+\alpha)^s} + \int_N^{+\infty} \frac{\mathrm{d}x}{(x+\alpha)^s} - \frac{1}{2(N+\alpha)^s} - s\int_N^{+\infty} \frac{\psi(x)}{(x+\alpha)^s}\mathrm{d}x\\
&= \sum_{n=0}^{N} \frac{1}{(n+\alpha)^s} + \frac{(N+\alpha)^{1-s}}{s-1} - \frac{1}{2(N+\alpha)^s} - s\int_N^{+\infty} \frac{\psi(x)}{(x+\alpha)^s}\mathrm{d}x.\end{aligned}$$

Notice that the last line converges uniformly in compact subsets of $\{s \in \mathbb{C} : s \ne 1, \sigma > 0\}$, so it is a valid expression for $\zeta(s,\alpha)$ the region $\eta \le \sigma \le 1, |t| \ge 2$. Now,

$$\begin{aligned}|\zeta(s,\alpha)| &\le \alpha^{-\sigma} + \sum_{n=1}^{N} n^{-\sigma} + \frac{(2N)^{1-\sigma}}{|t|} + \frac{1}{2}N^{-\sigma} + \frac{|s|}{2}\int_N^{+\infty} \frac{\mathrm{d}x}{(x+\alpha)^{\sigma+1}}\\
&\ll \alpha^{-\sigma} + N^{1-\sigma}\log N + N^{1-\sigma}|t|^{-1} + N^{-\sigma} + \frac{|t|}{\sigma}N^{-\sigma}.\end{aligned}$$

Finally, setting $N = \lfloor t \rfloor$, we win. □

Lemma 3.4. *Let $\eta > 0$ be fixed. Then for $(a, q) = 1$, $\eta \le \sigma \le 1$, and $|t| \ge 2$, one has*

$$D_k\left(s; \frac{a}{q}\right) \ll (q|t|^{(1-\sigma)} \log|t|)^k.$$

Proof. By Lemma 3.1 and Lemma 3.3, we have

$$\begin{aligned} D_k\left(s; \frac{a}{q}\right) &\ll q^{k(1-\sigma)}(q^\sigma + |t|^{1-\sigma}\log|t|)^k \\ &\ll (q + (q|t|)^{1-\sigma}\log|t|)^k \ll (q|t|^{(1-\sigma)}\log|t|)^k. \end{aligned}$$

□

To evaluate the residue term in the contour integration, we study $M_k(s; q)$ in details.

Lemma 3.5. *When q is squarefree, we have*

$$M_k(s; q) = \zeta(s)^k \Psi_k(s; q),$$

where

$$\Psi_k(s; q) = q \prod_{p|q} \left(1 - \frac{1}{p} - \left(1 - \frac{1}{p^s}\right)^k\right). \tag{3.4}$$

Proof. By definition (3.3), we know

$$\begin{aligned} M_k(s; q) &= \sum_{n \ge 1} \frac{\tau_k(n)}{n^s} \sum_{\substack{1 \le a \le q \\ (a,q)=1}} e\left(\frac{an}{q}\right) = \sum_{n\ge1} \frac{\tau_k(n) c_q(n)}{n^s} \\ &= \sum_{n\ge1} \frac{\tau_k(n)}{n^s} \sum_{d|(q,n)} d\mu\left(\frac{q}{d}\right) \\ &= \sum_{d|q} d^{1-s}\mu\left(\frac{q}{d}\right) \sum_{m\ge1} \frac{\tau_k(md)}{m^s}. \end{aligned} \tag{3.5}$$

By Euler product and the fact that d is squarefree, we know

$$\begin{aligned} \sum_{m\ge1} \frac{\tau_k(md)}{m^s} &= \prod_p \sum_{j\ge0} \frac{\tau_k(p^{j+\nu_p(d)})}{p^{js}} \\ &= \prod_{p\nmid d} \sum_{j\ge0} \frac{\tau_k(p^j)}{p^{js}} \prod_{p|d} \sum_{j\ge0} \frac{\tau_k(p^{j+1})}{p^{js}}. \end{aligned}$$

Note that $\tau_k(p^\ell)$ is the same as

$$\#\{(r_1, \dots, r_k) \in \mathbb{Z}_{\ge0} : r_1 + r_2 + \cdots + r_k = \ell\},$$

so we have

$$\sum_{j\geq 0}\frac{\tau_k(p^j)}{p^{js}}=\left(\sum_{r\geq 0}\frac{1}{p^{rs}}\right)^k=\left(1-\frac{1}{p^s}\right)^{-k}$$

and

$$\sum_{j\geq 0}\frac{\tau_k(p^{j+1})}{p^{js}}=p^s\left(\left(1-\frac{1}{p^s}\right)^{-k}-1\right).$$

Therefore, we have

$$\begin{aligned}\sum_{m\geq 1}\frac{\tau_k(md)}{m^s}&=\prod_{p\nmid d}\left(1-\frac{1}{p^s}\right)^{-k}\prod_{p|d}p^s\left(\left(1-\frac{1}{p^s}\right)^{-k}-1\right)\\&=\zeta(s)^k d^s\prod_{p|d}\left(1-\left(1-\frac{1}{p^s}\right)\right).\end{aligned}$$

Plugging this into (3.5), we obtain

$$\begin{aligned}M_k(s;q)&=\zeta(s)^k\mu(q)\sum_{d|q}d\mu(d)\prod_{p|d}\left(1-\left(1-\frac{1}{p^s}\right)\right)\\&=\zeta(s)^k\mu(q)\prod_{p|q}\left(1-p+p\left(1-\frac{1}{p^s}\right)^k\right)\\&=\zeta(s)^k q\prod_{p|q}\left(1-\frac{1}{p}-\left(1-\frac{1}{p^s}\right)^k\right)=\zeta(s)^k\Psi_k(s;q).\end{aligned}$$

□

Lemma 3.6. *For $|w|\leq\frac{1}{2}$, write*

$$\frac{(w\zeta(w+1))^k}{w+1}=\sum_{n\geq 0}\eta_k(n)w^n,$$

$$\Psi_k^{(h)}(q):=\left(\frac{\mathrm{d}}{\mathrm{d}w}\right)^h\Psi_k(w+1;q)\Bigg|_{w=0},$$

and

$$\theta_k^{(j)}(q)=\frac{1}{(k-1-j)!}\sum_{h=0}^{j}\frac{1}{h!}\eta_k(j-h)\Psi_k^{(h)}(q). \tag{3.6}$$

Then one has

$$\operatorname*{Res}_{s=1}\frac{x^s}{s}M_k(s;q)=x\sum_{j=0}^{k-1}\theta_k^{(j)}(q)(\log x)^{k-1-j},$$

$$\theta_k^{(0)}(q)=\frac{\varphi(q)}{(k-1)!}\prod_{p|q}\left(1-\left(1-\frac{1}{p}\right)^{k-1}\right),\quad \theta_k^{(j)}(q)\ll q^{\varepsilon}.$$

Proof. The residue calculation follows from

$$\begin{aligned}
\operatorname*{Res}_{s=1} \frac{x^s}{s} M_k(s;q) &= x \operatorname*{Res}_{w=0} \frac{x^w}{w^k} \frac{(w\zeta(w+1))^k}{w+1} \Psi_k(w+1;q) \\
&= x \operatorname*{Res}_{w=0} \frac{1}{w^k} \sum_{\ell,m\geq 0} \frac{(w\log x)^\ell}{\ell!} \eta_k(m) w^m \Psi_k(w+1;q) \\
&= x \sum_{\ell,m,n\geq 0} \frac{(\log x)^\ell \eta_k(m)}{\ell!} \operatorname*{Res}_{w=0} \frac{\Psi_k(w+1;q)}{w^{k-\ell-m}} \\
&= x \sum_{\substack{\ell,m\geq 0 \\ \ell+m\leq k-1}} \frac{(\log x)^\ell \eta_k(m) \Psi_k^{(k-1-\ell-m)}(q)}{\ell!(k-1-\ell-m)!} \\
&= x \sum_{0\leq m\leq j\leq k-1} \frac{(\log x)^{k-1-j} \eta_k(m) \Psi_k^{(j-m)}(q)}{(k-1-j)!(j-m)!} \\
&= x \sum_{0\leq h\leq j\leq k-1} \frac{(\log x)^{k-1-j} \eta_k(m) \Psi_k^{(h)}(q)}{(k-1-j)!h!} \\
&= x \sum_{j=0}^{k-1} (\log x)^{k-1-j} \underbrace{\sum_{h=0}^{j} \frac{\eta_k(j-h)\Psi_k^{(h)}(q)}{(k-1-j)!h!}}_{\theta_k^{(j)}(q)}.
\end{aligned}$$

The expression for $\theta_k^{(0)}(q)$ follows from (3.4) and $\eta_k(0) = 1$ that

$$\begin{aligned}
\theta_k^{(0)}(q) = \frac{1}{(k-1)!} \eta_k(0) \Psi_k(1;q) &= \frac{q}{(k-1)!} \prod_{p|q} \left(1 - \frac{1}{p} - \left(1 - \frac{1}{p}\right)^k\right) \\
= \frac{q}{(k-1)!} \prod_{p|q} \left(1 - \frac{1}{p}\right) \left(1 - \left(1 - \frac{1}{p}\right)^{k-1}\right) \\
= \frac{\varphi(q)}{(k-1)!} \prod_{p|q} \left(1 - \left(1 - \frac{1}{p}\right)^{k-1}\right).
\end{aligned}$$

To bound $\theta_k^{(j)}(q)$, it suffices to bound $\Psi_k^{(h)}(q)$. Observe that when $\sigma \geq 1 - \lambda$ and $\lambda > 0$,

$$|1 - p^{-1} - (1-p^{-s})^k| \leq \frac{1}{p} + \sum_{j=1}^{k} \binom{k}{j} p^{-j\sigma} \leq p^{-1} + (2^k - 1)p^{-\sigma} \leq 2^k p^{\lambda - 1},$$

so

$$|\Psi_k(s;q)| \leq q \prod_{p|q} \frac{2^k}{p^{1-\lambda}} = q^\lambda \tau_2(q)^{2^k}.$$

By the Cauchy integral formula, when $\lambda = (\log 2q)^{-1}$, one has

$$\begin{aligned}|\Psi_k^{(h)}(q)| &= \left| \frac{h!}{2\pi i} \oint_{|s-1|=\lambda} \frac{\Psi_k(s;q)}{(s-1)^{k+1}} \mathrm{d}s \right| \\ &\le h! \frac{q^\lambda \tau_2(q)^{2^k}}{\lambda^k} \le h! e \tau_2(q)^{2^k} (\log 2q)^k.\end{aligned}$$

Therefore, by (3.6), we have for $k \ge 2$ fixed and $0 \le j \le k-1$ that

$$|\theta_k^{(j)}(q)| \ll \tau_2(q)^{2^k} (\log 2q)^k \ll q^\varepsilon.$$

□

Now, we estimate $T_k(a/q;x)$, the $\beta = 0$ case of Proposition 1.6:

Proposition 3.1. *When $(a,q) = 1$, we have*

$$T_k\left(\frac{a}{q};x\right) = \frac{x}{\varphi(q)} \sum_{j=0}^{k-1} \theta_k^{(j)}(q)(\log x)^{k-1-j} + O(q^k x^{1-\frac{1}{k+1}+\varepsilon}).$$

Proof. Let $\eta > 0$ be fixed. Then by (3.1) and Cauchy's theorem, we have

$$\begin{aligned}T_k\left(\frac{a}{q};x\right) &= \operatorname*{Res}_{s=1} \frac{x^s}{s} D_k\left(s;\frac{a}{q}\right) + O(x^{1+\varepsilon}T^{-1}) \\ &\quad + \frac{1}{2\pi i}\left(\int_{c_0-iT}^{\eta-iT} + \int_{\eta-iT}^{\eta+iT} + \int_{\eta+iT}^{c_0+iT}\right) \frac{x^s}{s} D_k\left(s;\frac{a}{q}\right) \mathrm{d}s.\end{aligned}$$

By Lemma 3.2 and Lemma 3.6, we know

$$\begin{aligned}\operatorname*{Res}_{s=1} \frac{x^s}{s} D_k\left(s;\frac{a}{q}\right) &= \frac{1}{\varphi(q)} \operatorname*{Res}_{s=1} \frac{x^s}{s} M_k(s;q) \\ &= \frac{x}{\varphi(q)} \sum_{j=0}^{k-1} \theta_k^{(j)}(q)(\log x)^{k-1-j}.\end{aligned}$$

As for the remaining integrals, by Lemma 3.4, we know

$$\int_{\eta-iT}^{\eta+iT} \ll x^\eta q^k \int_1^T t^{k(1-\eta)-1}(\log t)^k \mathrm{d}t \ll q^k T^k x^{2\eta}.$$

$$\begin{aligned}\left|\int_{c_0-iT}^{\eta-iT}\right| + \left|\int_{\eta+iT}^{c_0+iT}\right| &\ll q^k T^{k-1} x^\eta \int_\eta^{c_0} \left(\frac{x}{T^k}\right)^\sigma \mathrm{d}x \\ &\ll q^k T^{k-1} x^\eta \left[\left(\frac{x}{T^k}\right)^\eta + \left(\frac{x}{T^k}\right)^{c_0}\right] \\ &\ll q^k T^{k-1} x^{2\eta} + q^k x^{1+\eta} T^{-1}.\end{aligned}$$

Setting $\eta = \frac{1}{2}\varepsilon$, we see that

$$T_k\left(\frac{a}{q};x\right) - \frac{x}{\varphi(q)}\sum_{j=0}^{k-1}\theta_k^{(j)}(q)(\log x)^{k-1-j}$$
$$\ll q^k(T^k + xT^{-1})x^{\varepsilon}.$$

Setting $T = x^{1/(k+1)}$, we obtain the desired result. □

Proof of Proposition 1.6. When $\beta \neq 0$, it follows from partial summation that

$$T_k\left(\frac{a}{q}+\beta;N\right) = \sum_{n=1}^{N} e(n\beta)\left[T_k\left(\frac{a}{q};n\right) - T_k\left(\frac{a}{q};n-1\right)\right]$$
$$= e(N\beta)T_k\left(\frac{a}{q};N\right) - \sum_{n=1}^{N-1}[e(n\beta) - e((n+1)\beta)]T_k\left(\frac{a}{q};n\right).$$

Therefore, the contribution from the main term in Proposition 3.1 is

$$\frac{1}{\varphi(q)}\sum_{n=1}^{N} e(n\beta)\sum_{j=0}^{k-1}\theta_k^{(j)}(q)\ x(\log x)^{k-1-j}\Big|_{n-1}^{n},$$

which is precisely $\varphi(q)^{-1}U_k(\beta;q)$ according (1.18) and (1.19).

Notice that

$$|e(n\beta) - e((n+1)\beta)| = \left|2\pi i\beta\int_{n+1}^{n} e(x\beta)\mathrm{d}x\right| \leq 2\pi|\beta|,$$

so the total contribution from the error term in Proposition 3.1 is

$$\ll q^k N^{1-\frac{1}{1+k}+\varepsilon} + q^k|\beta|N^{2-\frac{1}{1+k}+\varepsilon}.$$

□

3.2 Total contribution of major arcs

To compute the integral over $\mathfrak{M}$, we need information concerning the singular integral and the singular series.

Lemma 3.7 (Singular integral)**.** *When $q \leq \mathcal{L}^{K_2}$, we have*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} U_2(-\beta;q)U_k(\beta;q)S_0(\beta)\mathrm{d}\beta$$
$$= PN\sum_{j=0}^{k}\Xi_k^{(j)}(q)(\log N)^{k-j} + O(PN\mathcal{L}^{k+1-\frac{1}{2}K_1}),$$

where the numbers $\Xi_k^{(j)}(q)$ satisfies

$$\Xi_k^{(0)}(q) = \theta_2^{(0)}(q)\theta_k^{(0)}(q), \quad \Xi_k^{(j)}(q) \ll q^{\varepsilon}. \quad (0 \leq j \leq k)$$

Proof. By (1.18) and (1.17), we know

$$\begin{aligned}
&\int_{-\frac{1}{2}}^{\frac{1}{2}} U_2(-\beta;q)U_k(\beta;q)S_0(\beta)\mathrm{d}\beta \\
&\qquad = \sum_{\substack{n_1,n_2\le N\\ n_3\sim P\\ -n_1+n_2+n_3=0}} \Gamma_2(n_1;q)\Gamma_k(n_2;q) \\
&\qquad = \sum_{\substack{m\le N\\ n\sim P}} \Gamma_2(m+n;q)\Gamma_k(m;q).
\end{aligned}$$

Using (1.19), we have

$$\Gamma_2(n;q) = \theta_2^{(0)}(q)\log n + \theta_2^{(0)}(q) + \theta_2^{(1)}(q) + O\left(\frac{q^\varepsilon}{n}\right).$$

Therefore, we have

$$\begin{aligned}
&\sum_{\substack{m\le N\\ n\sim P}} \Gamma_2(m+n;q)\Gamma_k(m;q) \\
&\quad = \sum_{j=0}^{k-1} \theta_k^{(j)}(q)[\theta_2^{(0)}\Sigma_1^{(k-1-j)} + (\theta_2^{(0)}(q)+\theta_2^{(1)}(q))\Sigma_2^{(k-1-j)}] \\
&\quad + O(N\mathcal{L}).
\end{aligned} \tag{3.7}$$

in which

$$\Sigma_2^{(\ell)} = P\sum_{m\le N} x(\log x)^\ell|_{m-1}^m = PN(\log N)^\ell$$

and

$$\begin{aligned}
\Sigma_1^{(\ell)} &= \sum_{m\le N} x(\log x)^\ell|_{m-1}^m \sum_{n\sim P}\log(m+n) \\
&= \sum_{N\mathcal{L}^{-\frac{1}{2}K_1}<m\le N} x(\log x)^\ell|_{m-1}^m \sum_{n\sim P}\log(m+n) + O(N\mathcal{L}^{\ell+1-\frac{1}{2}K_1}).
\end{aligned}$$

When $N\mathcal{L}^{-\frac{1}{2}K_2} < m \le N$ and $n\sim P$, one has

$$\log(m+n) = \log m + \log\left(1+\frac{n}{m}\right) = \log m + O(\mathcal{L}^{-\frac{1}{2}K_1}),$$

so

$$\begin{aligned}
\Sigma_1^{(\ell)} &= \sum_{N\mathcal{L}^{-\frac{1}{2}K_1}<m\le N} x(\log x)^\ell|_{m-1}^m \log m + O(N\mathcal{L}^{\ell+1-\frac{1}{2}K_1}) \\
&= \sum_{m\le N} [(\log m)^{\ell+1} + \ell(\log m)^\ell] + O(N\mathcal{L}^{\ell+1-\frac{1}{2}K_1}) \\
&= \sum_{m\le N} [(\log m)^{\ell+1} + (\ell+1)(\log m)^\ell - (\log m)^\ell] + O(N\mathcal{L}^{\ell+1-\frac{1}{2}K_1}) \\
&= \sum_{m\le N} [x(\log x)^{\ell+1}|_{m-1}^m - (\log m)^\ell] + O(N\mathcal{L}^{\ell+1-\frac{1}{2}K_1}) \\
&= N(\log N)^{\ell+1} - \sum_{m\le N} (\log m)^\ell + O(N\mathcal{L}^{\ell+1-\frac{1}{2}K_1}) \\
&= N\left[(\log N)^{\ell+1} - \sum_{r=0}^{\ell} c_{\ell,r}(\log N)^r\right] + O(N\mathcal{L}^{\ell+1-\frac{1}{2}K_1}),
\end{aligned}$$

where

$$c_{\ell,r} = \frac{(-1)^{\ell-r}\ell!}{r!}.$$

Plugging these into (3.7), we get

$$\begin{aligned}
&\sum_{\substack{m\le N\\ n\sim P}} \Gamma_2(m+n;q)\Gamma_k(m;q) \\
&\quad = PN\sum_{j=0}^{k-1}\theta_2^{(0)}(q)\theta_k^{(j)}(q)\left[(\log N)^{k-j} - \sum_{r=0}^{k-j-1} c_{k-1-j,r}(\log N)^r\right] \\
&\quad + PN(\theta_2^{(0)}(q)+\theta_2^{(1)}(q))\sum_{j=0}^{k-1}\theta_k^{(j)}(q)(\log N)^{k-1-j} \\
&\quad + O(PN\mathcal{L}^{k+1-\frac{1}{2}K_1}).
\end{aligned}$$

Combining like terms, we obtain the desired conclusion. □

Lemma 3.8 (Singular series)**.** *Define the singular series*

$$\xi_k(j) = \sum_{q\ge 1}\frac{\mu(q)}{\varphi(q)^2}\Xi_k^{(j)}(q). \tag{3.8}$$

Then

$$\sum_{q\le\mathcal{L}^{K_2}}\frac{\mu(q)}{\varphi(q)^2}\Xi_k^{(j)}(q) = \xi_k(j) + O(\mathcal{L}^{-\frac{1}{2}K_2})$$

and

$$\xi_k(0) = \frac{1}{(k-1)!}\prod_{p|q}\left(1-\frac{1}{p}+\frac{1}{p}\left(1-\frac{1}{p}\right)^{k-1}\right).$$

Proof. The first conclusion comes from $\Xi_k^{(j)}(q) \ll q^{\varepsilon}, \varphi(q) \gg q^{1-\varepsilon}$ and

$$\sum_{q>\mathcal{L}^{K_2}} \frac{\mu(q)}{\varphi(q)^2} \Xi_k^{(j)}(q) \ll \sum_{q>\mathcal{L}^{K_2}} \frac{1}{q^{\frac{3}{2}}} \ll \mathcal{L}^{-\frac{1}{2}K_2}.$$

The second conclusion follows from (1.20):

$$\begin{aligned}
\xi_k(0) &= \sum_{q\geq 1} \frac{\mu(q)}{\varphi(q)^2} \theta_2^{(0)}(q)\theta_k^{(0)}(q) \\
&= \frac{1}{(k-1)!} \sum_{q\geq 1} \frac{\mu(q)}{q} \prod_{p|q} \left(1 - \left(1 - \frac{1}{p}\right)^{k-1}\right) \\
&= \frac{1}{(k-1)!} \prod_{p|q} \left(1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^{k-1}\right).
\end{aligned}$$

□

Now, we estimate the integral over $\mathfrak{M}(a/q)$. Recall from (1.16) that

$$\mathfrak{M}(a/q) = \left[\frac{a}{q} - \frac{\mathcal{L}^{K_2}}{N}, \frac{a}{q} + \frac{\mathcal{L}^{K_2}}{N}\right].$$

Combining Proposition 1.6, Proposition 1.5, and Lemma 2.1, we see that when $\alpha = \frac{a}{q} + \beta \in \mathfrak{M}(a/q)$, one has

$$\begin{aligned}
T_2(-\alpha; N)T_k(\alpha; N)S(\alpha) = \frac{\mu(q)}{\varphi(q)^3} U_2(-\beta; q)U_k(\beta; q)S_0(\beta) \\
+ O(PN^2 e^{-\frac{1}{2}c\sqrt{\mathcal{L}}}).
\end{aligned} \tag{3.9}$$

Due to the presence of $\mu(q)$, (3.9) is valid for all $q \leq \mathcal{L}^{K_2}$ and $(a, q) = 1$. Therefore, we have

$$\begin{aligned}
&\int_{\mathfrak{M}(a/q)} T_2(-\alpha; N)T_k(\alpha; N)S(\alpha)\mathrm{d}\alpha \\
&= \frac{\mu(q)}{\varphi(q)^3} \int_{-\mathcal{L}^{K_2}/N}^{\mathcal{L}^{K_2}/N} U_2(-\beta; q)U_k(\beta; q)S_0(\beta)\mathrm{d}\beta + O(PNe^{-\frac{1}{4}c\sqrt{\mathcal{L}}}).
\end{aligned} \tag{3.10}$$

To estimate this integral by the *singular integral* we want to replace the range of integration to $[-\frac{1}{2}, \frac{1}{2}]$, so estimating the error requires bounds for $U_2(-\beta; q)U_k(\beta; q)S_0(\beta)$. When $|\beta| \leq \frac{1}{2}$, $S_0(\beta)$ satisfies

$$|S_0(\beta)| = \left|\sum_{n\sim P} e(n\beta)\right| \leq \frac{2}{|\beta|}.$$

By (1.18) and (1.19), the task for $U_k(\beta;q)$ is reduced to studying

$$\sum_{n=1}^{N} e(n\beta)x(\log x)^{\ell}|_{n-1}^{n}. \tag{3.11}$$

By the mean value theorem, one has

$$x(\log x)^{\ell}|_{n-1}^{n} = (\log n)^{\ell} + \ell(\log n)^{\ell-1} + O\left(\frac{(\log n)^{\ell-1}}{n}\right).$$

Now, by partial summation,

$$\sum_{n=1}^{N} e(n\beta)(\log n)^{j} = \sum_{m=1}^{N} e(m\beta)(\log N)^{j} - \sum_{n=1}^{N-1}\left(\sum_{m=1}^{n} e(m\beta)\right)(\log x)^{j}|_{n}^{n+1},$$

so

$$\left|\sum_{n=1}^{N} e(n\beta)(\log n)^{j}\right| \le \frac{4}{|\beta|}(\log N)^{j}.$$

Conclusively, (3.11) becomes

$$\sum_{n=1}^{N} e(n\beta)x(\log x)^{\ell}|_{n-1}^{n} \ll |\beta|^{-1}\mathcal{L}^{\ell} + \frac{\mathcal{L}^{\ell-1}}{N},$$

which is $\ll |\beta|^{-1}\mathcal{L}^{\ell}$ in the range $|\beta| \le \mathcal{L}^{K_2}N^{-1}$. Combining this with (1.18), (1.19), Proposition 1.6, and $q \le \mathcal{L}^{K_2}$, we have

$$U_k(\beta;q) \ll |\beta|^{-1}\mathcal{L}^{k},$$

so

$$U_2(-\beta;q)U_k(\beta;q)S_0(\beta) \ll |\beta|^{-3}\mathcal{L}^{2+k}.$$

Consequently, when $K_2 \ge 2 + k + K_1$,

$$\begin{aligned}
&\int_{-\mathcal{L}^{K_2}/N}^{\mathcal{L}^{K_2}/N} U_2(-\beta;q)U_k(\beta;q)S_0(\beta)\mathrm{d}\beta \\
&\qquad = \int_{-\frac{1}{2}}^{\frac{1}{2}} U_2(-\beta;q)U_k(\beta;q)S_0(\beta)\mathrm{d}\beta + O\left(\mathcal{L}^{2k}\int_{\mathcal{L}^{K_2}/N}^{+\infty}\beta^{-3}\mathrm{d}\beta\right) \\
&\qquad = \int_{-\frac{1}{2}}^{\frac{1}{2}} U_2(-\beta;q)U_k(\beta;q)S_0(\beta)\mathrm{d}\beta + O(PN\mathcal{L}^{-K_2}).
\end{aligned}$$

By Lemma 3.7, this becomes

$$= PN\sum_{j=0}^{k} \Xi_k^{(j)}(q)(\log N)^{k-j} + O(PN\mathcal{L}^{k+1-\frac{1}{2}K_1}) + O(PN\mathcal{L}^{-K_2}).$$

Assume $K_1 \geq 4k+4$, so plugging this into (3.10), one has

$$\int_{\mathfrak{M}(a/q)} T_2(-\alpha;N)T_k(\alpha;N)S(\alpha)\mathrm{d}\alpha$$
$$= PN\sum_{j=0}^{k}\frac{\mu(q)}{\varphi(q)^3}\Xi_k^{(j)}(q)(\log N)^{k-j} + O(PN\mathcal{L}^{-\frac{1}{4}K_1}) + O(PN\mathcal{L}^{-K_2}). \tag{3.12}$$

Recall from (1.16) that

$$\mathfrak{M} = \bigcup_{\substack{1\leq a\leq q\leq \mathcal{L}^{K_2}\\ (a,q)=1}} \mathfrak{M}(a/q),$$

so when $K_2 \geq 4k$, it follows from (3.12) and Lemma 3.8 that the integral over $\mathfrak{M}$ is

$$\int_{\mathfrak{M}} T_2(-\alpha;N)T_k(\alpha;N)S(\alpha)\mathrm{d}\alpha$$
$$= \sum_{q\leq \mathcal{L}^{K_2}}\sum_{\substack{1\leq a\leq q\\ (a,q)=1}}\int_{\mathfrak{M}(a/q)} T_2(-\alpha;N)T_k(\alpha;N)S(\alpha)\mathrm{d}\alpha$$
$$= \sum_{j=0}^{k}\left(\sum_{q\leq \mathcal{L}^{K_2}}\frac{\mu(q)}{\varphi(q)^2}\Xi_k^{(j)}(q)\right)(\log N)^{k-j} + O(PN\mathcal{L}^{-\frac{1}{4}\min(K_1,K_2)})$$
$$= PN\sum_{j=0}^{k}\xi_k(j)(\log N)^{k-j} + O(PN\mathcal{L}^{-\frac{1}{4}\min(K_1,K_2)}).$$

3.3 Minor arc estimates

For $\alpha \in \mathfrak{m}$, it follows from Dirichlet's approximation theorem that one can always find some $q \leq N\mathcal{L}^{-K_2}$ and $(a,q)=1$ such that

$$\left|\alpha - \frac{a}{q}\right| < \frac{\mathcal{L}^{K_2}}{qN}.$$

Because α cannot lie in $\mathfrak{M}(a/q)$, we must have $q > \mathcal{L}^{K_2}$. Combining this with Proposition 1.7, one has

$$\sup_{\alpha\in\mathfrak{m}}|S(\alpha)| \ll (P\mathcal{L}^{-\frac{1}{2}K_2} + P^{\frac{1}{2}}N^{\frac{1}{2}-\frac{1}{2}K_2})\mathcal{L}^4 \ll P\mathcal{L}^{4+\frac{1}{2}K_1-\frac{1}{2}K_2}.$$

Therefore, by Lemma 2.1, we have

$$\left|\int_{\mathfrak{m}} T_2(-\alpha;N)T_k(\alpha;N)S(\alpha)\mathrm{d}\alpha\right| \leq \sup_{\alpha\in\mathfrak{m}}|S(\alpha)|\int_0^1 |T_2(-\alpha;N)T_k(\alpha;N)|\mathrm{d}\alpha$$
$$\ll P\mathcal{L}^{4+\frac{1}{2}K_1-\frac{1}{2}K_2}\int_0^1(|T_2(-\alpha;N)|^2+|T_k(\alpha;N)|^2)\mathrm{d}\alpha$$
$$= P\mathcal{L}^{4+\frac{1}{2}K_1-\frac{1}{2}K_2}\sum_{n\leq N}(\tau_2(n)^2+\tau_k(n)^2) \ll PN\mathcal{L}^{k^2+3+\frac{1}{2}K_1-\frac{1}{2}K_2}.$$

3.4 Asymptotic formula for the number of solutions

Proof. Proof of Proposition 1.8 Combining the results from §3.2 and §3.3, we deduce that when $K_1, K_2 \gg_k 1$,

$$X_k(N) = PN\sum_{j=0}^{k}\xi_k(j)(\log N)^{k-j}$$
$$+ O(PN\mathcal{L}^{-\frac{1}{4}\min(K_1,K_2)}) + O(PN\mathcal{L}^{k^2+3+\frac{1}{2}K_1-\frac{1}{2}K_2}).$$

Now, if we require $K_1 \geq 2k^2+6$ and $K_2 = 4K_1$, then the error term reduces to

$$\ll PN\mathcal{L}^{-\frac{1}{4}K_1} + PN\mathcal{L}^{K_1-2K_1} \ll PN\mathcal{L}^{-\frac{1}{4}K_1}.$$

Finally, renaming $K_1 = 4K$, we win. □

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